Name______________________________ Section____________

MATH 152 Honors FINAL EXAM Spring 2014

Sections 201-202 Solutions P. Yasskin

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Total	/106

Multiple Choice: (14 problems, 4 points each)

1. Find the area under the curve $y = \dfrac{1}{x^2+1}$ above the interval $[0,1]$.

a. $\dfrac{1}{4}$

b. $\dfrac{\pi}{4}$ CORRECT

c. $\dfrac{1}{2}$

d. 1

e. $\dfrac{\pi}{2}$

Solution: $A = \displaystyle\int_0^1 \frac{1}{x^2+1}\,dx = \Big[\arctan x\Big]_0^1 = \arctan 1 - \arctan 0 = \frac{\pi}{4}$

2. The region under the curve $y = \dfrac{1}{x^2+1}$ above the interval $[0,1]$ is revolved about the y-axis. Find the volume of the resulting solid.

a. $\pi \ln(2)$ CORRECT

b. $\pi \ln(2) - \pi$

c. $2\pi \ln(2)$

d. $4\pi \ln(2)$

e. $4\pi \ln(2) - 4\pi$

Solution: $A = \displaystyle\int_0^1 2\pi r h\,dx \qquad r = x \qquad h = \frac{1}{x^2+1} \qquad u = x^2+1 \qquad du = 2x\,dx$

$$= \int_0^1 2\pi x \frac{1}{x^2+1}\,dx = \int \pi \frac{1}{u}\,du = \pi \ln u = \Big[\pi \ln(x^2+1)\Big]_0^1 = \pi \ln(2)$$

3. A plate with constant density ρ has the shape of the region below $y = \dfrac{1}{x^2+1}$ above the interval $[0,1]$. Find the x-coordinate of its center of mass.

a. $\dfrac{\pi}{2}\ln 2$

b. $\dfrac{2}{\pi}\ln 2$ CORRECT

c. $\dfrac{\pi}{2\ln 2}$

d. $\dfrac{2}{\rho\ln 2}$

e. $\dfrac{\rho}{2}\ln 2$

Solution: $M = \rho A = \rho\dfrac{\pi}{4}$ $\quad M_1 = \displaystyle\int_0^1 \frac{\rho x}{x^2+1}\,dx = \left[\frac{\rho}{2}\ln(x^2+1)\right]_0^1 = \frac{\rho}{2}\ln 2$

$$\bar{x} = \frac{M_1}{M} = \frac{\rho\ln 2}{2}\,\frac{4}{\rho\pi} = \frac{2}{\pi}\ln 2$$

4. Compute $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta}\,d\theta$.

a. $-\infty$

b. -1

c. $\dfrac{1}{2}$

d. 1

e. ∞ CORRECT

Solution: $\displaystyle\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta}\,d\theta = \int_0^{\pi/4} \csc^2\theta\,d\theta = \big[-\cot\theta\big]_0^{\pi/4} = -\cot\frac{\pi}{4} + \lim_{\theta\to 0^+}\cot\theta = \infty$

OR:

$u = \tan\theta$

$du = \sec^2\theta\,d\theta$

$$\int_0^{\pi/4} \frac{\sec^2\theta}{\tan^2\theta}\,d\theta = \int \frac{du}{u^2} = \left[\frac{-1}{u}\right] = \left[\frac{-1}{\tan\theta}\right]_0^{\pi/4} = \frac{-1}{\tan(\pi/4)} + \lim_{\theta\to 0^+}\frac{1}{\tan\theta} = \infty$$

5. Compute $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$. HINT: $\sin(2\theta) = 2\sin\theta\cos\theta$

a. $\dfrac{1}{2}\arcsin x - \dfrac{x}{3}(1-x^2)^{3/2} + C$

b. $\dfrac{1}{2}\arcsin x + x\sqrt{1-x^2} + C$

c. $\dfrac{1}{2}\arcsin x - x\sqrt{1-x^2} + C$

d. $\dfrac{1}{2}\arcsin x - \dfrac{x}{2}\sqrt{1-x^2} + C$ CORRECT

e. $\dfrac{1}{2}\arcsin x + \dfrac{x}{2}\sqrt{1-x^2} + C$

Solution:

$x = \sin\theta$

$dx = \cos\theta\,d\theta$

$$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int \sin^2\theta\,d\theta = \int \frac{1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C = \frac{1}{2}\left(\arcsin x - x\sqrt{1-x^2}\right) + C$$

6. Compute $\int (\ln x)^2 dx$. HINT: $u = (\ln x)^2$.

a. $x\ln^2 x - x^2 \ln x - \frac{1}{2}x^2 + C$

b. $x\ln^2 x - x^2 \ln x + \frac{1}{2}x^2 + C$

c. $x\ln^2 x - 2x\ln x - 2x + C$

d. $x\ln^2 x - 2x\ln x + 2x + C$ CORRECT

e. $x\ln^2 x - 2x\ln x + 4x + C$

Solution:

$u = (\ln x)^2 \qquad dv = dx$

$du = 2(\ln x)\frac{1}{x}\,dx \qquad v = x$

$$\int (\ln x)^2 dx = x(\ln x)^2 - 2\int \ln x\,dx = x(\ln x)^2 - 2(x\ln x - x) + C$$

7. Which of the following is the general partial fraction expansion of $\dfrac{4x^2+5}{(x-2)^2(x^2+9)^2}$?

a. $\dfrac{A}{(x-2)^2} + \dfrac{Bx+C}{(x^2+9)^2}$

b. $\dfrac{Ax+B}{(x-2)^2} + \dfrac{Cx+D}{(x^2+9)^2}$

c. $\dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{(x^2+9)} + \dfrac{Ex+F}{(x^2+9)^2}$ CORRECT

d. $\dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2} + \dfrac{Cx}{(x^2+9)} + \dfrac{Dx}{(x^2+9)^2}$

e. $\dfrac{A}{(x-2)} + \dfrac{Bx+C}{(x-2)^2} + \dfrac{D}{(x^2+9)} + \dfrac{Ex+F}{(x^2+9)^2}$

Solution: For each linear factor to the p, you need a linear denominator for each power up to p. For each quadratic factor to the p, you need a quadratic denominator for each power up to p. The linear denominators get a constant on top. The quadratic denominators get a linear on top. So the correct expansion is: $\dfrac{4x^2+5}{(x-2)^2(x^2+9)^2} = \dfrac{A}{(x-2)} + \dfrac{B}{(x-2)^2} + \dfrac{Cx+D}{(x^2+9)} + \dfrac{Ex+F}{(x^2+9)^2}$

8. The curve $x = y^3$ between $y = 0$ and $y = 1$ is rotated about the y-axis. Find the area of the resulting surface.

a. $\frac{\pi}{27}(10^{3/2} - 1)$ CORRECT

b. $48\pi(10^{3/2} - 1)$

c. $\frac{7}{27}\pi$

d. 336π

e. $48\pi 10^{3/2}$

Solution: $A = \int_{y=0}^{1} 2\pi r\,ds = \int_0^1 2\pi x\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \int_0^1 2\pi y^3\sqrt{(3y^2)^2 + 1}\,dy$

$= \int_0^1 2\pi y^3\sqrt{9y^4+1}\,dy \qquad u = 9y^4 + 1 \qquad du = 36y^3\,dy \qquad \frac{1}{36}\,du = y^3 dy$

$= \frac{\pi}{18}\int_1^{10}\sqrt{u}\,du = \left[\frac{\pi}{18}\frac{2u^{3/2}}{3}\right]_1^{10} = \frac{\pi}{27}(10^{3/2} - 1)$

9. Solve the differential equation $\frac{dy}{dx} = \frac{y}{x}$ with $y(1) = 3$. Then $y(3) =$

a. $\frac{1}{3}$

b. 1

c. 3

d. 6

e. 9 CORRECT

Solution: Separate: $\frac{dy}{y} = \frac{dx}{x}$ $\int \frac{dy}{y} = \int \frac{dx}{x}$ $\ln y = \ln x + C$

Use the initial condition: $\ln 3 = \ln 1 + C$ $C = \ln 3$

Substitute back: $\ln y = \ln x + \ln 3$ Solve: $y = e^{\ln x + \ln 3} = 3x$ So: $y(3) = 9$

10. Solve the differential equation $x\frac{dy}{dx} = y + x^2$ with $y(1) = 2$. Then $y(2) =$

a. $\frac{1}{2}$

b. 2

c. $\frac{13}{6}$

d. 3

e. 6 CORRECT

Solution: Standard form: $\frac{dy}{dx} - \frac{1}{x}y = x$ Integrating factor: $I = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

Multiply by I: $\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 1$ Rewrite: $\frac{d}{dx}\left(\frac{1}{x}y\right) = 1$

Integrate: $\frac{y}{x} = \int 1\,dx = x + C$ Use the initial condition: $\frac{2}{1} = 1 + C$ $C = 1$

Substitute back: $\frac{y}{x} = x + 1$ Solve: $y = x^2 + x$ So: $y(2) = 6$

11. A ball is dropped from 27 feet and bounces to $\frac{2}{3}$ of its previous height on each bounce. Find the total length travelled during an infinite number of bounces.

a. 54

b. 81

c. 108

d. 135 CORRECT

e. 162

Solution: $L = 27 + 2\sum_{n=1}^{\infty} 27\left(\frac{2}{3}\right)^n = 27 + 54\frac{\frac{2}{3}}{1 - \frac{2}{3}} = 27 + 54\frac{2}{3-2} = 27 + 108 = 135$

12. Compute $\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3}$

a. $\frac{1}{3}$ CORRECT

b. $\frac{1}{6}$

c. 0

d. ∞

e. $-\infty$

Solution:

$$\lim_{x \to 0} \frac{\sin x - x\cos x}{x^3} = \lim_{x \to 0} \frac{\left(x - \frac{x^3}{6} + \cdots\right) - x\left(1 - \frac{x^2}{2} + \cdots\right)}{x^3} = \lim_{x \to 0} \frac{-\frac{x^3}{6} + \frac{x^3}{2} + \cdots}{x^3} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

13. Compute $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n+1)!}$

a. $\frac{\sqrt{3}}{2}$

b. $\frac{3\sqrt{3}}{2}$ CORRECT

c. $\frac{\sqrt{3}}{6}$

d. $\frac{3}{2}$

e. 0

Solution: $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sin x$ So

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{9^n (2n+1)!} = 3\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\pi}{3}\right)^{2n+1} = 3\sin\frac{\pi}{3} = \frac{3\sqrt{3}}{2}$$

14. Find the center and radius of the sphere $x^2 - 4x + y^2 + z^2 + 6z + 4 = 0$

a. center: $(-2, 0, 3)$ radius: $R = 2$

b. center: $(2, 0, -3)$ radius: $R = 3$ CORRECT

c. center: $(-2, 0, 3)$ radius: $R = 3$

d. center: $(2, 0, -3)$ radius: $R = 9$

e. center: $(-2, 0, 3)$ radius: $R = 9$

Solution: $(x^2 - 4x + 4) + y^2 + (z^2 + 6z + 9) - 9 = 0$ $(x-2)^2 + y^2 + (z+3)^2 = 9$

center: $(2, 0, -3)$ radius: $R = 3$

Work Out (3 questions, Points indicated)

Show all you work.

15. (22 points) Consider the series $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

a. (4) Determine whether the series is absolutely convergent, convergent but not absolutely convergent or divergent.

Solution:

Method 1: Apply the Ratio Test

$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{2^n} \frac{2^{n-1}}{n} \right| = \frac{1}{2} < 1$ So the series is absolutely convergent.

Method 2: The related absolute series is $\sum_{n=1}^{\infty} \frac{n}{2^{n-1}}$.

For $n \geq 12$, $2^{n-1} > n^3$ So $\frac{1}{2^{n-1}} < \frac{1}{n^3}$ $\frac{n}{2^{n-1}} < \frac{1}{n^2}$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p = 2 > 1$) the absolute series converges by the comparison test.

b. (1) Compute S_7, the 7^{th} partial sum for S. Do not simplify.

Solution: $S_7 = \sum_{n=1}^{7} \frac{(-1)^{n-1} n}{2^{n-1}} = 1 - \frac{2}{2} + \frac{3}{4} - \frac{4}{8} + \frac{5}{16} - \frac{6}{32} + \frac{7}{64}$

c. (3) Find a bound on the remainder $|R_7| = |S - S_7|$ when S_7 is used to approximate S. Name the theorem you used.

Solution: $|R_7| < |a_8| = \frac{8}{2^7} = \frac{8}{128} = \frac{1}{16}$ by the alternating series bound theorem

d. (1 Real easy) Find a power series $S(x)$ centered at 0 whose value at $x = \frac{-1}{2}$ is the given series S, i.e $S\left(\frac{-1}{2}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

Solution: $S(x) = \sum_{n=1}^{\infty} n x^{n-1}$ So $S\left(\frac{-1}{2}\right) = \sum_{n=1}^{\infty} n \left(\frac{-1}{2}\right)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

#15 continued. Recall $S = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{2^{n-1}}$.

e. (10) Find the interval of convergence of the power series $S(x)$ from part (d). Give the radius and check the endpoints.

Solution: $S(x) = \sum_{n=1}^{\infty} nx^{n-1}$ Apply the ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| = |x|\lim_{n\to\infty}\left|\frac{(n+1)}{n}\right| = |x| < 1 \qquad \text{radius: } R = 1$$

Endpoint $x = 1$: The series becomes $\sum_{n=1}^{\infty} n$ which diverges by the nth term divergence test since $\lim_{n\to\infty} n = \infty \neq 0$

Endpoint $x = -1$: The series becomes $\sum_{n=1}^{\infty} (-1)^{n-1} n$ which diverges by the nth term divergence test since $\lim_{n\to\infty} (-1)^{n-1} n$ diverges.

So the interval of convergence is $(-1, 1)$.

f. (2) Find a function $f(x)$ whose Maclaurin series is the power series $S(x)$ from part (d).

Solution:

$$S(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=1}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(1-x) - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2} = f(x)$$

g. (1) Use $f(x)$ to find the sum of the series S.

Solution: Since $S(x) = \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, we have

$$S = S\left(\frac{-1}{2}\right) = \frac{1}{\left(1 - \frac{-1}{2}\right)^2} = \frac{4}{9}$$

16. (20 points) Let $f(x) = \ln(x)$.

a. (6) Find the Taylor series for $f(x)$ centered at $x = 4$.

Solution:

$f(x) = \ln(x)$ $\qquad f(4) = \ln(4)$

$f'(x) = \frac{1}{x}$ $\qquad f'(4) = \frac{1}{4}$

$f''(x) = \frac{-1}{x^2}$ $\qquad f''(4) = \frac{-1}{4^2}$

$f'''(x) = \frac{2}{x^3}$ $\qquad f'''(4) = \frac{2}{4^3}$

$f^{(4)}(x) = \frac{-3!}{x^4}$ $\qquad f^{(4)}(4) = \frac{-3!}{4^4}$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$ $\qquad f^{(n)}(4) = \frac{(-1)^{n+1}(n-1)!}{4^n}$

$$T(x) = \ln 4 + \sum_{n=1}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n$$
$$= \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)!}{4^n n!}(x-4)^n$$
$$= \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4^n n}(x-4)^n$$

b. (11) The Taylor series for $f(x)$ centered at $x = 3$ is $T(x) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(x-3)^n$.

Find the interval of convergence for the Taylor series centered at $x = 3$
Give the radius and check the endpoints.

Solution: Apply the Ratio Test:

$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-3|^{n+1}}{3^{n+1}(n+1)} \frac{3^n n}{|x-3|^n} = \frac{|x-3|}{3} < 1 \qquad |x-3| < 3$ radius: $R = 3$.

endpoint: $x = 0$: $\quad T(0) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(-3)^n = \ln 3 - \sum_{n=1}^{\infty} \frac{1}{n}$

diverges (harmonic series)

endpoint: $x = 6$: $\quad T(6) = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3^n n}(3)^n = \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

converges (alternating series test)

Interval of Convergence: $(0, 6]$

c. (3) If the cubic Taylor polynomial centered at $x = 3$ is used to approximate $\ln(x)$ on the interval $[2, 5]$, use the Taylor's Inequality to bound the error.

Taylor's Inequality:
Let $T_n(x)$ be the n^{th}-degree Taylor polynomial for $f(x)$ centered at $x = a$ and let $R_n(x) = f(x) - T_n(x)$ be the remainder. Then
$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$
provided $M \ge |f^{(n+1)}(c)|$ for all c between a and x.

Solution: Here $n = 3$ and $a = 3$. Since c is between 3 and x, while x can be anything in $[2, 5]$, then c can be anything in $[2, 5]$.

$|f^{(4)}(c)| = \left|\frac{6}{c^4}\right|$ is largest on $[2, 5]$ when $x = 2$. So we take $M = \frac{6}{2^4} = \frac{3}{8}$

For x in $[2, 5]$, the largest value of $|x-3|$ is $|5-3| = 2$.

So $|R_3(x)| \le \frac{M}{4!}|x-4|^4 = \frac{3}{8 \cdot 4!}|x-4|^4 \le \frac{3}{8 \cdot 4!}2^4 = \frac{1}{4}$

17. (4 points) When a ball with mass, m, is dropped from a height, h, and falls under the forces of gravity with acceleration, g, and air resistance with drag coefficient, k, the altitude, $y(t)$, satisfies the differential equation,

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}$$

The solution is

$$y(t) = h - \frac{m^2}{k^2}g\,e^{-\frac{kt}{m}} + \frac{m^2}{k^2}g - \frac{m}{k}g\,t$$

Verify that this solution reduces to the standard freefall formula (no air resistance) by taking the **limit** of the solution as k approaches 0. (DO NOT SOLVE ANY DIFFERENTIAL EQUATIONS.)

Solution: We use the Maclaurin series $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots$ with $x = -\frac{kt}{m}$:

$$\begin{aligned}
\lim_{k\to 0} y(t) &= \lim_{k\to 0}\left(h - \frac{m^2}{k^2}g\,e^{-\frac{kt}{m}} + \frac{m^2}{k^2}g - \frac{m}{k}g\,t\right) \\
&= \lim_{k\to 0}\left(h - \frac{m^2}{k^2}g\left[1 - \frac{kt}{m} + \frac{1}{2}\left(\frac{kt}{m}\right)^2 - \frac{1}{6}\left(\frac{kt}{m}\right)^3 + \cdots\right] + \frac{m^2}{k^2}g - \frac{m}{k}g\,t\right) \\
&= \lim_{k\to 0}\left(h - \frac{m^2}{k^2}g\left[\frac{1}{2}\left(\frac{kt}{m}\right)^2 - \frac{1}{6}\left(\frac{kt}{m}\right)^3 + \cdots\right]\right) \\
&= \lim_{k\to 0}\left(h - \frac{1}{2}gt^2 + \frac{1}{6}g\frac{kt^3}{m} + \cdots\right) = h - \frac{1}{2}gt^2
\end{aligned}$$

18. (4 points) The curve $x = y^2$ for $0 \le y \le \sqrt{2}$ is revolved around the x-axis. Find the area of the resulting surface.

Solution: $\frac{dx}{dy} = 2y$

$$A = \int_0^{\sqrt{2}} 2\pi r\,ds = \int_0^{\sqrt{2}} 2\pi y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}\,dy = \int_0^{\sqrt{2}} 2\pi y\sqrt{4y^2 + 1}\,dy$$

Let $u = 4y^2 + 1$, $du = 8y\,dy$ $\frac{1}{8}du = y\,dy$

$$A = \frac{1}{8}\int_1^9 2\pi\sqrt{u}\,du = \left[\frac{1}{4}\frac{2u^{3/2}}{3}\right]_1^9 = \frac{1}{6}(9^{3/2} - 1) = \frac{1}{6}(26) = \frac{13}{3}$$