

Name _____ Section _____
 MATH 152H FINAL EXAM Spring 2016
 Sections 201-202 Solutions P. Yasskin

1-13	/52
14	/20
15	/20
16	/5
17	/5
18	/5
Total	/107

Multiple Choice: (13 problems, 4 points each)

1.

Average Value of a Function

New Problem or Modify or Make Your Own Problem

Find the average value of the function $f(x) = \frac{2}{9}x^2$ on the interval $[a,b] = [0,3]$.

- a. $\frac{1}{6}$
- b. $\frac{1}{3}$
- c. $\frac{2}{3}$ correct choice
- d. $\frac{4}{3}$
- e. 2

Solution: $f_{ave} = \frac{1}{b-a} \int_a^b f(x) dx = \frac{1}{3} \int_0^3 \frac{2}{9}x^2 dx = \frac{2}{27} \frac{x^3}{3} \Big|_0^3 = \frac{2}{3}$

2.

Integrals Which are Improper at an Endpoint

New Problem

Problem Statement:
 Determine if the following improper integral is convergent or divergent.

$$\int_0^2 \frac{x}{(x-2)^3} dx$$

If convergent, compute it.
 If divergent, determine if it is +infinity, -infinity, or neither.

- a. converges to $\frac{1}{4}$
- b. converges to $-\frac{1}{4}$
- c. diverges to $-\infty$ correct choice
- d. diverges to ∞
- e. diverges but not to $\pm\infty$

Solution: $u = x - 2 \quad du = dx \quad x = u + 2$

$$\int_0^2 \frac{x}{(x-2)^3} dx = \int_{-2}^0 \frac{u+2}{u^3} du = \left[-\frac{1}{u} - \frac{1}{u^2} \right]_{-2}^0 = \lim_{u \rightarrow 0^-} \left(\frac{-u-1}{u^2} \right) - \left(\frac{2-1}{4} \right) = \frac{-1}{(0^-)^2} = -\infty$$

3.

Integration By Parts

Indefinite Integral Use integration by parts

Definite Integral to compute the integral:

$$J = \int_1^2 \frac{\ln(x)}{x^2} dx$$

- a. $\frac{3 - \ln(2)}{2}$
- b. $\frac{\ln(2) - 3}{2}$
- c. $\frac{\ln(2) - 1}{2}$
- d. $\frac{-\ln(2)}{2}$
- e. $\frac{1 - \ln(2)}{2}$ correct choice

Solution:

$u = \ln(x)$	$dv = \frac{1}{x^2} dx$
$du = \frac{1}{x} dx$	$v = \frac{-1}{x}$

$$J = \left[\frac{-\ln(x)}{x} - \int \frac{-1}{x^2} dx \right]_1^2 = \left[\frac{-\ln(x)}{x} - \frac{1}{x} \right]_1^2$$

$$= \left(\frac{-\ln(2)}{2} - \frac{1}{2} \right) - \left(\frac{-\ln(1)}{1} - \frac{1}{1} \right) = \frac{-\ln(2)}{2} + \frac{1}{2}$$

4.

Trigonometric Integrals

Indefinite Integral Use a substitution

Definite Integral to compute the integral:

$$J = \int_0^{\frac{1}{3}\pi} \sec^4 \theta d\theta$$

- a. $-2\sqrt{3}$
- b. $2\sqrt{3}$ correct choice
- c. $-\frac{27}{5}$
- d. $\frac{27}{5}$
- e. $\frac{81}{5}\sqrt{3}$

Solution:

$u = \tan \theta$
$du = \sec^2 \theta d\theta$

$$J = \int_0^{\pi/3} (1 + \tan^2 \theta) \sec^2 \theta d\theta = \int_0^{\sqrt{3}} (1 + u^2) du = \left[u + \frac{u^3}{3} \right]_0^{\sqrt{3}}$$

$$= \left(\sqrt{3} + \frac{\sqrt{3}^3}{3} \right) = 2\sqrt{3}$$

5.

Integration by Trigonometric Substitution

New Integral

Goal: Evaluate the indefinite integral using a trigonometric substitution:

$$I = \int \frac{1}{x^2 \sqrt{(36x^2 - 1)}} dx$$

- a. $\int \frac{36 \cos^3 \theta d\theta}{\sin \theta}$
 b. $\int \frac{\cos^3 \theta d\theta}{36 \sin \theta}$
 c. $\frac{1}{6} \int \cos \theta d\theta$
 d. $\int \cos \theta d\theta$
 e. $6 \int \cos \theta d\theta$ correct choice

Simply identify the integral after the substitution.

Solution:

$6x = \sec \theta$
$x = \frac{1}{6} \sec \theta$
$dx = \frac{1}{6} \sec \theta \tan \theta d\theta$

$$I = \int \frac{\frac{1}{6} \sec \theta \tan \theta d\theta}{\left(\frac{1}{6} \sec \theta\right)^2 \sqrt{\sec^2 \theta - 1}} = \int \frac{6 \sec \theta \tan \theta d\theta}{\sec^2 \theta \tan \theta} = \int \frac{6 d\theta}{\sec \theta}$$

$$= 6 \int \cos \theta d\theta$$

6.

Partial Fractions: Finding Coefficients

New Function Include Completing the Square

Goal: Find the coefficients in the partial fraction expansion:

$$\frac{3x^2 - 3x - 2}{x^3(x+2)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x^3} + \frac{A_4}{x+2}$$

- a. $A_2 = -\frac{1}{2}$ $A_3 = -2$
 b. $A_2 = \frac{1}{2}$ $A_3 = -2$
 c. $A_2 = -1$ $A_3 = -1$ correct choice
 d. $A_2 = 1$ $A_3 = 1$
 e. $A_2 = 1$ $A_3 = -1$

Just find A_2 and A_3 .

Solution: Clear the denominator:

$$3x^2 - 3x - 2 = A_1 x^2(x+2) + A_2 x(x+2) + A_3(x+2) + A_4 x^3 \quad (*)$$

Plug in $x = 0$: $-2 = A_3(2)$ $A_3 = -1$

Differentiate (*): $6x - 3 = A_1(3x^2 + 4x) + A_2(2x + 2) + A_3 + A_4(3x^2)$

Plug in $x = 0$: $-3 = A_2(2) + A_3 = 2A_2 - 1$ $A_2 = -1$

7.

Volume By Slicing

New Problem or Modify or Make Your Own Problem Quit

Find the volume of the solid whose base is a semi-circle of radius 4 with the diameter edge parallel to the y axis, and whose cross sections perpendicular to the y direction are squares.

- a. $\frac{8}{3}$
- b. $\frac{128}{3}$
- c. $\frac{256}{3}$ correct choice
- d. 8π
- e. $\frac{128}{3}\pi$

Solution: The base is shown. It is an y-integral.

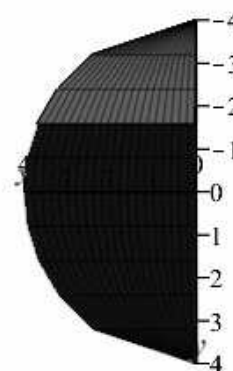
The horizontal slices have width $s = \sqrt{16 - y^2}$.

So the area of the square cross sections is $A = s^2 = 16 - y^2$.

So the volume is

$$V = \int_{-4}^4 A(y) dy = \int_{-4}^4 (16 - y^2) dy$$

$$= \left[16y - \frac{y^3}{3} \right]_{-4}^4 = 2 \left(64 - \frac{64}{3} \right) = \frac{256}{3}$$



8.

Direction Fields

New Differential Equation Quit

Problem Statement:

Find the direction field of the differential equation: $\frac{d}{dx}y(x) = x - y$

Select a Plot

Plot # 1

Plot # 2

Plot # 3

Plot # 4

(a.)

(b.)

(c.)

(d.)

correct choice

Solution: The slope is zero (0) when $x = y$, which only happens in (b).

9. **Arc Length of a Curve in 2D**

Functions of t or

Find the arc length of the curve $x = 2t^2$, $y = 1/3t^3$, between $t = 0$ and $t = 3$.

a. $\frac{31}{3}$
 b. $\frac{61}{3}$ correct choice
 c. $\frac{64}{3}$
 d. $\frac{122}{3}$
 e. $\frac{125}{3}$

Solution:
$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \int_0^3 \sqrt{(4t)^2 + (t^2)^2} dt$$

$$= \int_0^3 t\sqrt{16+t^2} dt = \left[\frac{(16+t^2)^{3/2}}{3} \right]_0^3 = \frac{125}{3} - \frac{64}{3} = \frac{61}{3}$$

10. The function $y = \frac{x}{x^2 + 1}$ is a solution to which differential equation?

- a. $\frac{dy}{dx} = \frac{y}{x^3 + x} + y^2$
 b. $\frac{dy}{dx} = \frac{y}{x^3 + x} - y^2$ correct choice
 c. $\frac{dy}{dx} = \frac{y^2}{x^3 + x} + y^2$
 d. $\frac{dy}{dx} = \frac{y^2}{x^3 + x} - y^2$
 e. $\frac{dy}{dx} = -\frac{y}{x^3 + x} + y^2$

Solution:
$$\frac{dy}{dx} = \frac{(x^2 + 1) - x2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2}$$

$$\frac{y}{x^3 + x} = \frac{1}{x^3 + x} \frac{x}{x^2 + 1} = \frac{1}{(x^2 + 1)^2}$$

$$y^2 = \frac{x^2}{(x^2 + 1)^2}$$

$$\frac{y^2}{x^3 + x} = \frac{1}{x^3 + x} \frac{x^2}{(x^2 + 1)^2} = \frac{x}{(x^2 + 1)^3}$$

So $\frac{dy}{dx} = \frac{y}{x^3 + x} - y^2$

11. **Geometric Series**

Goal: Compute the sum of the geometric series (if the sum exists).

$$S = \sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} -1 \left(-\frac{2}{9}\right)^n$$

a. $-\frac{9}{7}$
 b. $-\frac{9}{11}$
 c. $-\frac{4}{63}$
 d. $-\frac{4}{99}$ correct choice
 e. diverges

Solution: $a = -1\left(-\frac{2}{9}\right)^2 = -\frac{4}{81}$ $r = -\frac{2}{9}$ $S = \frac{a}{1-r} = \frac{-\frac{4}{81}}{1 + \frac{2}{9}} = -\frac{4}{99}$

12.

Computing Limits Using Maclaurin Series

Goal: Use a Maclaurin Series to evaluate this limit:

$L = \lim_{x \rightarrow 0} \frac{\cos(x^2) - 1}{x^6}$

New Limit

- a. 0
- b. $\frac{1}{2}$
- c. $\frac{1}{3}$
- d. $\frac{1}{24}$
- e. diverges correct choice

Solution: $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} \dots$ $\cos(x^2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} \dots$

$$L = \lim_{x \rightarrow 0} \frac{1 - \frac{x^4}{2!} + \frac{x^8}{4!} \dots - 1}{x^6} = \lim_{x \rightarrow 0} \left(-\frac{x^{-2}}{2!} + \frac{x^2}{4!} \dots \right) = -\infty$$

13.

Separable Differential Equations

New Differential Equation or Modify or Make Your Own Problem

Find a General Solution Solve an Initial Value Problem

Find the solution $y = F(x)$ of the differential equation $\frac{dy}{dx} = -y^2 x^3$ satisfying the initial condition $y(1) = 2$

- a. $y = -\frac{4}{x^4} + 6$
- b. $y = \frac{4}{x^4} - 2$
- c. $y = \frac{4}{x^4 + 1}$ correct choice
- d. $y = \frac{4}{x^4 - 12}$
- e. $y = \frac{4}{x^4} + \frac{3}{4}$

Solution: Separate: $\int \frac{dy}{y^2} = \int -x^3 dx$ Integrate: $-\frac{1}{y} = -\frac{x^4}{4} + C$

Plug in the initial condition, $x = 1$ when $y = 2$: $-\frac{1}{2} = -\frac{1}{4} + C \Rightarrow C = -\frac{1}{4}$

Substitute for C and solve for y : $-\frac{1}{y} = -\frac{x^4}{4} - \frac{1}{4}$ $\frac{1}{y} = \frac{x^4 + 1}{4}$ $y = \frac{4}{x^4 + 1}$

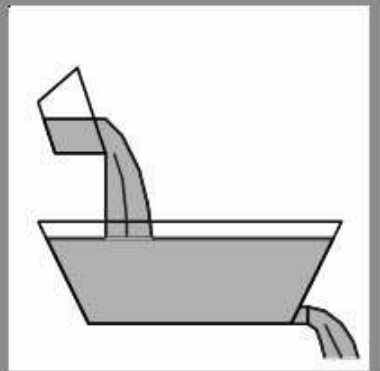
Work Out (5 questions, Points indicated. Show all you work.)

14. (20 points)

Mixing Problem [New Problem] [Quit]

Problem Statement:

A tank is initially filled with 650 L of sugar water with a concentration of 30 gm/L. A bucket is pouring sugar water of concentration 40 gm/L into the tank at a rate of 2 L/min. The solution is kept well mixed, and drains at a rate of 2 L/min. Find the total amount of sugar, $S(t)$, dissolved in the tank at time t . Then find the concentration of sugar in the tank at time $t = 60$ min.



a. (8 pts) Write the differential equation and initial condition for $S(t)$.

Solution: $\frac{dS}{dt} \frac{\text{gm}}{\text{min}} = 40 \frac{\text{gm}}{\text{L}} \cdot 2 \frac{\text{L}}{\text{min}} - \frac{S \text{gm}}{650 \text{L}} \cdot 2 \frac{\text{L}}{\text{min}}$ or $\frac{dS}{dt} = 80 - \frac{1}{325} S$

$S(0) = 30 \frac{\text{gm}}{\text{L}} 650 \text{L} = 19500 \text{gm}$

b. (9 pts) Solve the initial value problem for $S(t)$.

Solution Method 1: The equation is separable. Separate and integrate.

$$\int \frac{dS}{80 - \frac{1}{325} S} = \int dt \quad -325 \ln \left| 80 - \frac{1}{325} S \right| = t + C$$

Solve: $\ln \left| 80 - \frac{1}{325} S \right| = -\frac{t}{325} - \frac{C}{325} \quad \left| 80 - \frac{1}{325} S \right| = e^{-C/325} e^{-t/325}$
 $80 - \frac{1}{325} S = A e^{-t/325} \quad \text{where } A = \pm e^{-C/325} \quad S = 26000 - 325 A e^{-t/325}$

Use the initial condition: At $t = 0$, we have $S = 19500$. So

$19500 = 26000 - 325A, \quad A = 20$

Substitute back: $S = 26000 - 6500 e^{-t/325}$

Solution Method 2: The equation is linear: Put it in standard form:

$$\frac{dS}{dt} + \frac{1}{325} S = 80$$

Find the integrating factor: $I = \exp\left(\int \frac{1}{325} dt\right) = e^{t/325}$

Multiply the standard equation by I : $e^{t/325} \frac{dS}{dt} + \frac{1}{325} e^{t/325} S = 80 e^{t/325}$

Recognize the left side as the derivative of a product: $\frac{d}{dt}(e^{t/325} S) = 80 e^{t/325}$

Integrate and solve: $e^{t/325} S = \int 80 e^{t/325} dt = 80 \cdot 325 e^{t/325} + C \quad S = 26000 + C e^{-t/325}$

Use the initial condition: At $t = 0$, we have $S = 19500$. So

$$19500 = 26000 + C \quad C = -6500$$

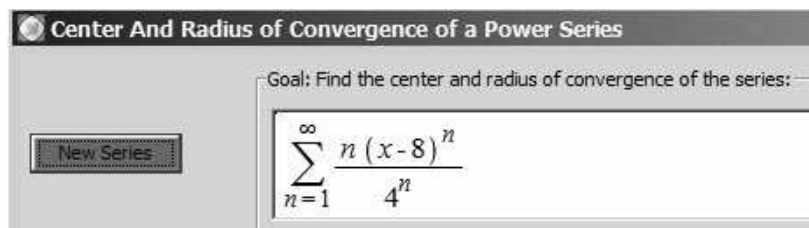
$$\text{Substitute back: } S = 26000 - 6500e^{-t/325}$$

- c. (3 pts) Find the concentration in the tank at $t = 60$ min.

$$\text{Solution: } S = 26000 - 6500e^{-60/325} = 26000 - 6500e^{-12/65}$$

$$\text{So the concentration is } \frac{S}{650} = \frac{26000 - 6500e^{-12/65}}{650} = 40 - 10e^{-12/65}$$

15. (20 points)



Also find the interval of convergence by checking the endpoints.

- a. (2 pts) Identify the center: $a = \underline{\quad 8 \quad}$

- b. (8 pts) Find the radius of convergence:

Solution: Apply the Ratio Test:

$$\rho = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \frac{(n+1)|x-8|^{n+1}}{4^{n+1}} \cdot \frac{4^n}{n|x-8|^n} = \frac{|x-8|}{4} \lim_{n \rightarrow \infty} \frac{(n+1)}{n} = \frac{|x-8|}{4} < 1$$

$$|x-8| < 4 \quad 4 < x < 12 \quad R = \underline{\quad 4 \quad}$$

- c. (8 pts) Check the endpoints:

Solution:

$$x = 4: \quad \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n \quad \lim_{n \rightarrow \infty} (-1)^n n = \text{divergent} \neq 0$$

Diverges by the n^{th} -Term Divergence Test

$$x = 12: \quad \sum_{n=1}^{\infty} \frac{n(4)^n}{4^n} = \sum_{n=1}^{\infty} n \quad \lim_{n \rightarrow \infty} n = \infty \neq 0$$

Diverges by the n^{th} -Term Divergence Test

- d. (2 pts) Summarize the interval of convergence: $I = \underline{\quad (4, 12) \quad}$

16. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is absolutely convergent, convergent but not absolutely or divergent. Explain all tests you use.

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent by the Alternating Series Test since the $(-1)^n$ says it is alternating, $\frac{1}{n^{1/3}}$ is decreasing and $\lim_{n \rightarrow \infty} \frac{1}{n^{1/3}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is divergent because it is a p -series with

$$p = \frac{1}{3} < 1.$$

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent but not absolutely.

17. (5 points) The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$ converges by the Integral Test.

If it is approximated by its 100th partial sum S_{100} , compute the integral bound on the error in this approximation.

Solution: The bound is

$$|E_7| = |S - S_{100}| < \int_{100}^{\infty} \frac{1}{n^2 + 1} dn = \left[\arctan(n) \right]_{100}^{\infty} = \frac{\pi}{2} - \arctan(100) \quad (\approx 0.01)$$

18. (5 points) Compute the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 3^{2n+1}}$.

Solution: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)! 3^{2n+1}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$