Name______________________________ Section___________

MATH 152H FINAL EXAM Spring 2016

Sections 201-202 Solutions P. Yasskin

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Total	/107

Multiple Choice: (13 problems, 4 points each)

1. **Average Value of a Function**

New Problem or Modify or Make Your Own Problem

Find the average value of the function f(x) = 2/9*x^2 on the interval [a,b] = [0,3].

a. $\frac{1}{6}$

b. $\frac{1}{3}$

c. $\frac{2}{3}$ correct choice

d. $\frac{4}{3}$

e. 2

Solution: $f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx = \frac{1}{3}\int_0^3 \frac{2}{9}x^2\,dx = \frac{2}{27}\frac{x^3}{3}\Big|_0^3 = \frac{2}{3}$

2. **Integrals Which are Improper at an Endpoint**

New Problem

Problem Statement:

Determine if the following improper integral is convergent or divergent.

$$\int_0^2 \frac{x}{(x-2)^3}\,dx$$

If convergent, compute it.

If divergent, determine if it is + infinity, - infinity, or neither.

a. converges to $\frac{1}{4}$

b. converges to $-\frac{1}{4}$

c. diverges to $-\infty$ correct choice

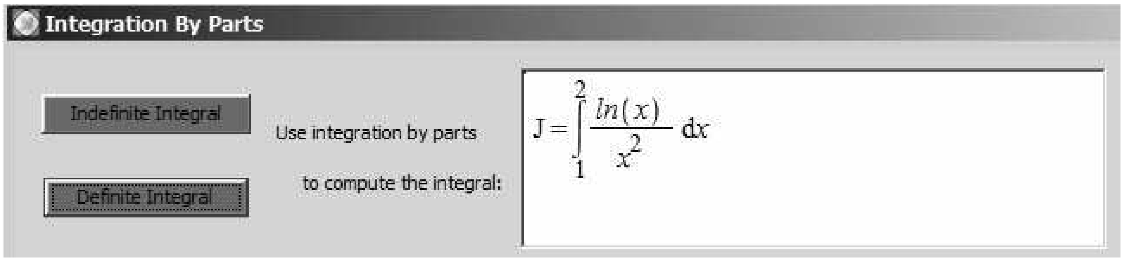
d. diverges to ∞

e. diverges but not to $\pm\infty$

Solution: $u = x-2 \qquad du = dx \qquad x = u+2$

$$\int_0^2 \frac{x}{(x-2)^3}\,dx = \int_{-2}^0 \frac{u+2}{u^3}\,du = \left[-\frac{1}{u} - \frac{1}{u^2}\right]_{-2}^0 = \lim_{u\to 0^-}\left(\frac{-u-1}{u^2}\right) - \left(\frac{2-1}{4}\right) = \frac{-1}{(0^-)^2} = -\infty$$

3.



a. $\frac{3 - \ln(2)}{2}$

b. $\frac{\ln(2) - 3}{2}$

c. $\frac{\ln(2) - 1}{2}$

d. $\frac{-\ln(2)}{2}$

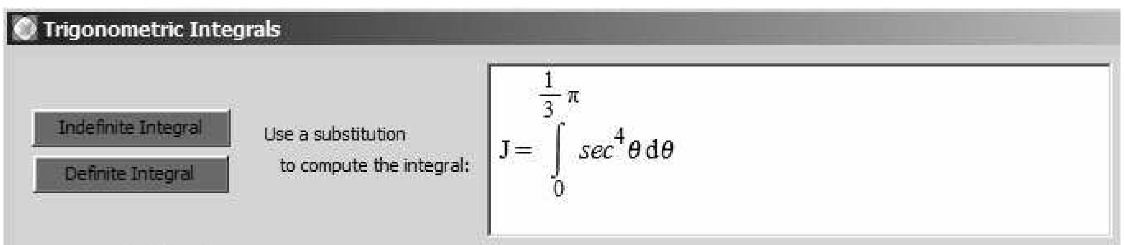
e. $\frac{1 - \ln(2)}{2}$ correct choice

Solution:

$u = \ln(x)$	$dv = \frac{1}{x^2}dx$
$du = \frac{1}{x}dx$	$v = \frac{-1}{x}$

$$J = \left[\frac{-\ln(x)}{x} - \int \frac{-1}{x^2}dx\right]_1^2 = \left[\frac{-\ln(x)}{x} - \frac{1}{x}\right]_1^2$$
$$= \left(\frac{-\ln(2)}{2} - \frac{1}{2}\right) - \left(\frac{-\ln(1)}{1} - \frac{1}{1}\right) = \frac{-\ln(2)}{2} + \frac{1}{2}$$

4.



a. $-2\sqrt{3}$

b. $2\sqrt{3}$ correct choice

c. $-\frac{27}{5}$

d. $\frac{27}{5}$

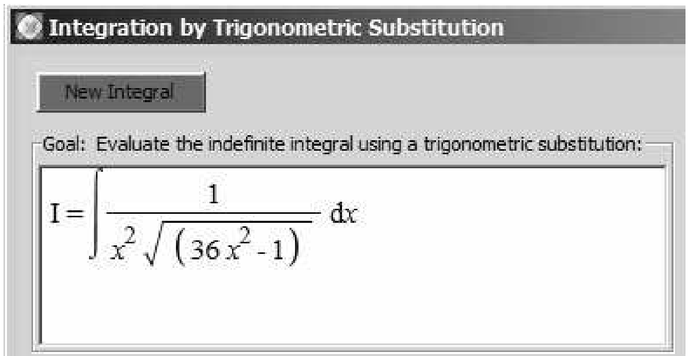
e. $\frac{81}{5}\sqrt{3}$

Solution:

$u = \tan\theta$
$du = \sec^2\theta\,d\theta$

$$J = \int_0^{\pi/3}(1 + \tan^2\theta)\sec^2\theta\,d\theta = \int_0^{\sqrt{3}}(1 + u^2)\,du = \left[u + \frac{u^3}{3}\right]_0^{\sqrt{3}}$$
$$= \left(\sqrt{3} + \frac{\sqrt{3}^3}{3}\right) = 2\sqrt{3}$$

5.



a. $\int \frac{36\cos^3\theta \, d\theta}{\sin\theta}$

b. $\int \frac{\cos^3\theta \, d\theta}{36\sin\theta}$

c. $\frac{1}{6}\int \cos\theta \, d\theta$

d. $\int \cos\theta \, d\theta$

e. $6\int \cos\theta \, d\theta$ correct choice

Simply identify the integral after the substitution.

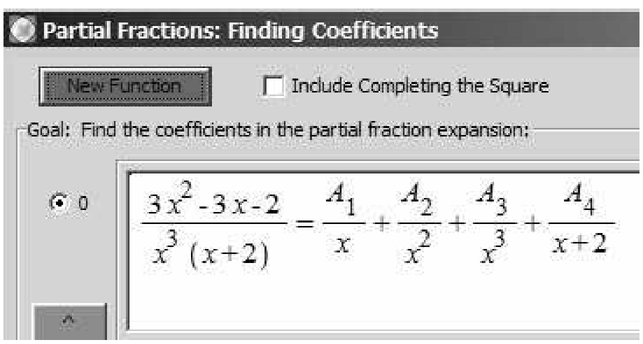
Solution:

$6x = \sec\theta$
$x = \frac{1}{6}\sec\theta$
$dx = \frac{1}{6}\sec\theta\tan\theta \, d\theta$

$$I = \int \frac{\frac{1}{6}\sec\theta\tan\theta \, d\theta}{\left(\frac{1}{6}\sec\theta\right)^2\sqrt{\sec^2\theta - 1}} = \int \frac{6\sec\theta\tan\theta \, d\theta}{\sec^2\theta\tan\theta} = \int \frac{6 \, d\theta}{\sec\theta}$$

$$= 6\int \cos\theta \, d\theta$$

6.



a. $A_2 = -\frac{1}{2} \quad A_3 = -2$

b. $A_2 = \frac{1}{2} \quad A_3 = -2$

c. $A_2 = -1 \quad A_3 = -1$ correct choice

d. $A_2 = 1 \quad A_3 = 1$

e. $A_2 = 1 \quad A_3 = -1$

Just find A_2 and A_3.

Solution: Clear the denominator:

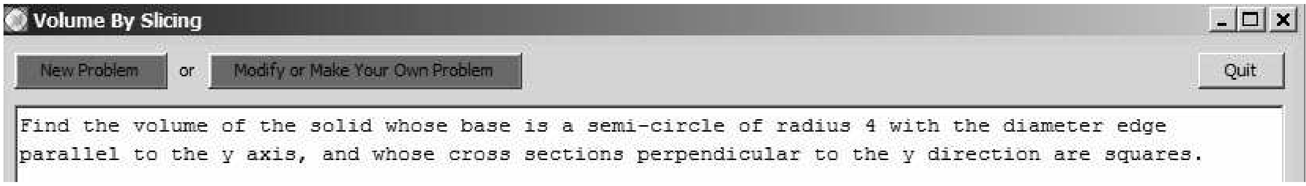
$3x^2 - 3x - 2 = A_1x^2(x+2) + A_2x(x+2) + A_3(x+2) + A_4x^3$ (*)

Plug in $x = 0$: $-2 = A_3(2)$ $A_3 = -1$

Differentiate (*): $6x - 3 = A_1(3x^2 + 4x) + A_2(2x + 2) + A_3 + A_4(3x^2)$

Plug in $x = 0$: $-3 = A_2(2) + A_3 = 2A_2 - 1$ $A_2 = -1$

7.



a. $\dfrac{8}{3}$

b. $\dfrac{128}{3}$

c. $\dfrac{256}{3}$ correct choice

d. 8π

e. $\dfrac{128}{3}\pi$

Solution: The base is shown. It is an y-integral.

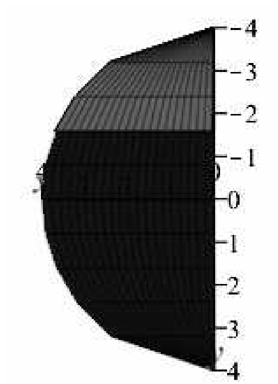
The horizontal slices have width $s = \sqrt{16 - y^2}$.

So the area of the square cross sections is $A = s^2 = 16 - y^2$.

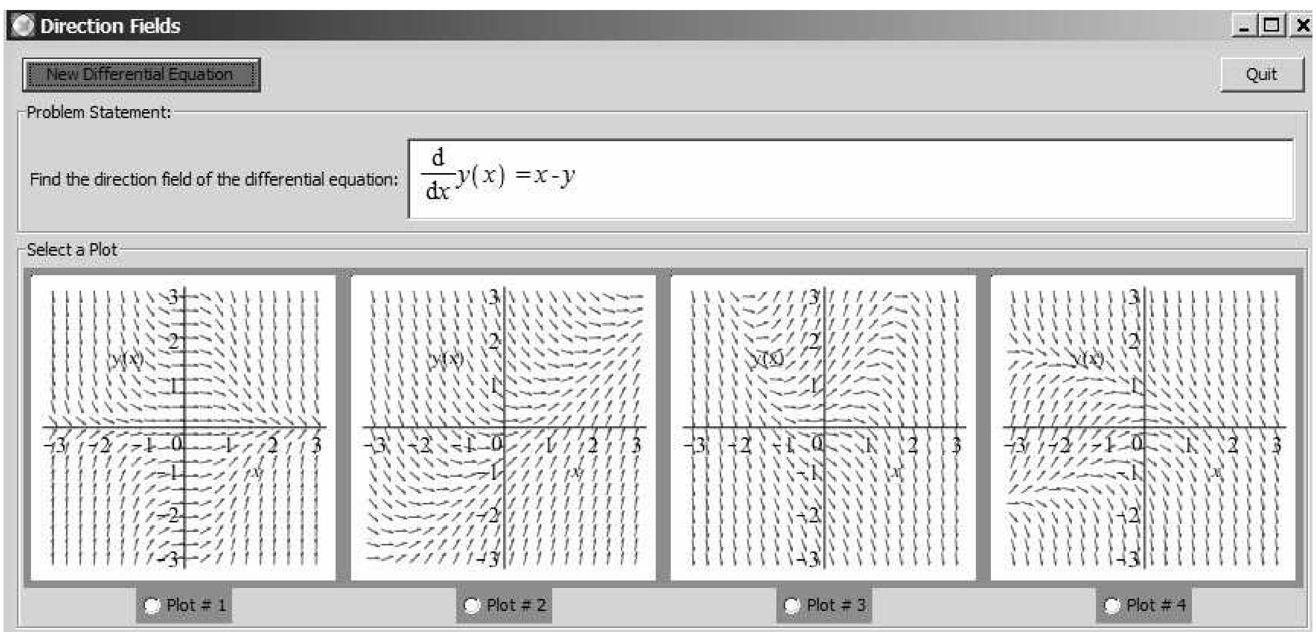
So the volume is

$$V = \int_{-4}^{4} A(y)\,dy = \int_{-4}^{4} (16 - y^2)\,dy$$

$$= \left[\, 16y - \frac{y^3}{3} \,\right]_{-4}^{4} = 2\left(64 - \frac{64}{3}\right) = \frac{256}{3}$$



8.

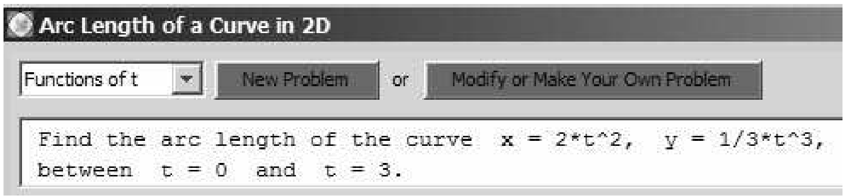


(**a**.) (**b**.) (**c**.) (**d**.)

correct choice

Solution: The slope is zero (0) when $x = y$, which only happens in (b).

9.



a. $\frac{31}{3}$

b. $\frac{61}{3}$ correct choice

c. $\frac{64}{3}$

d. $\frac{122}{3}$

e. $\frac{125}{3}$

Solution: $L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^3 \sqrt{(4t)^2 + (t^2)^2}\, dt$

$$= \int_0^3 t\sqrt{16 + t^2}\, dt = \left[\frac{(16+t^2)^{3/2}}{3}\right]_0^3 = \frac{125}{3} - \frac{64}{3} = \frac{61}{3}$$

10. The function $y = \frac{x}{x^2+1}$ is a solution to which differential equation?

a. $\frac{dy}{dx} = \frac{y}{x^3+x} + y^2$

b. $\frac{dy}{dx} = \frac{y}{x^3+x} - y^2$ correct choice

c. $\frac{dy}{dx} = \frac{y^2}{x^3+x} + y^2$

d. $\frac{dy}{dx} = \frac{y^2}{x^3+x} - y^2$

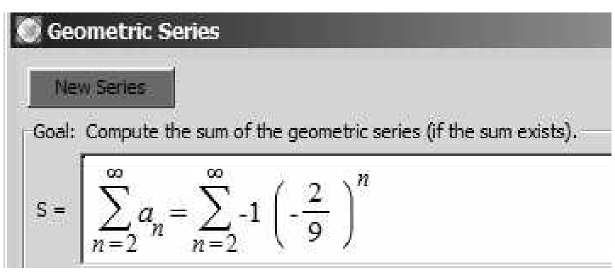
e. $\frac{dy}{dx} = -\frac{y}{x^3+x} + y^2$

Solution: $\frac{dy}{dx} = \frac{(x^2+1) - x2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$ $\qquad y^2 = \frac{x^2}{(x^2+1)^2}$

$\frac{y}{x^3+x} = \frac{1}{x^3+x}\frac{x}{x^2+1} = \frac{1}{(x^2+1)^2}$ $\qquad \frac{y^2}{x^3+x} = \frac{1}{x^3+x}\frac{x^2}{(x^2+1)^2} = \frac{x}{(x^2+1)^3}$

So $\frac{dy}{dx} = \frac{y}{x^3+x} - y^2$

11.



a. $-\frac{9}{7}$

b. $-\frac{9}{11}$

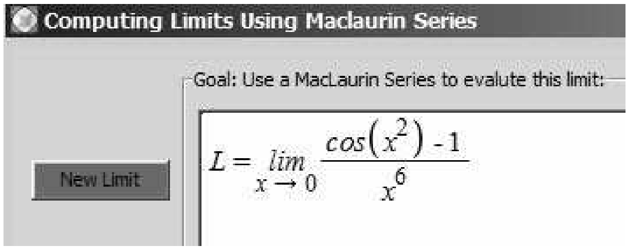
c. $-\frac{4}{63}$

d. $-\frac{4}{99}$ correct choice

e. diverges

Solution: $a = -1\left(-\frac{2}{9}\right)^2 = -\frac{4}{81}$ $\qquad r = -\frac{2}{9}$ $\qquad S = \frac{a}{1-r} = \frac{-\frac{4}{81}}{1+\frac{2}{9}} = -\frac{4}{99}$

12.



a. 0

b. $\frac{1}{2}$

c. $\frac{1}{3}$

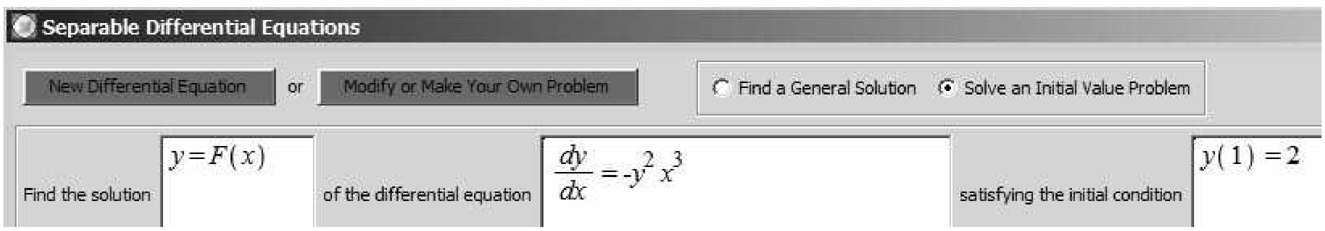
d. $\frac{1}{24}$

e. diverges correct choice

Solution: $\cos(u) = 1 - \frac{u^2}{2!} + \frac{u^4}{4!} \cdots \qquad \cos(x2) = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} \cdots$

$$L = \lim_{x \to 0} \frac{1 - \frac{x^4}{2!} + \frac{x^8}{4!} \cdots - 1}{x^6} = \lim_{x \to 0} \left(-\frac{x^{-2}}{2!} + \frac{x^2}{4!} \cdots \right) = -\infty$$

13.



a. $y = -\frac{4}{x^4} + 6$

b. $y = \frac{4}{x^4} - 2$

c. $y = \frac{4}{x^4 + 1}$ correct choice

d. $y = \frac{4}{x^4 - 12}$

e. $y = \frac{4}{x^4} + \frac{3}{4}$

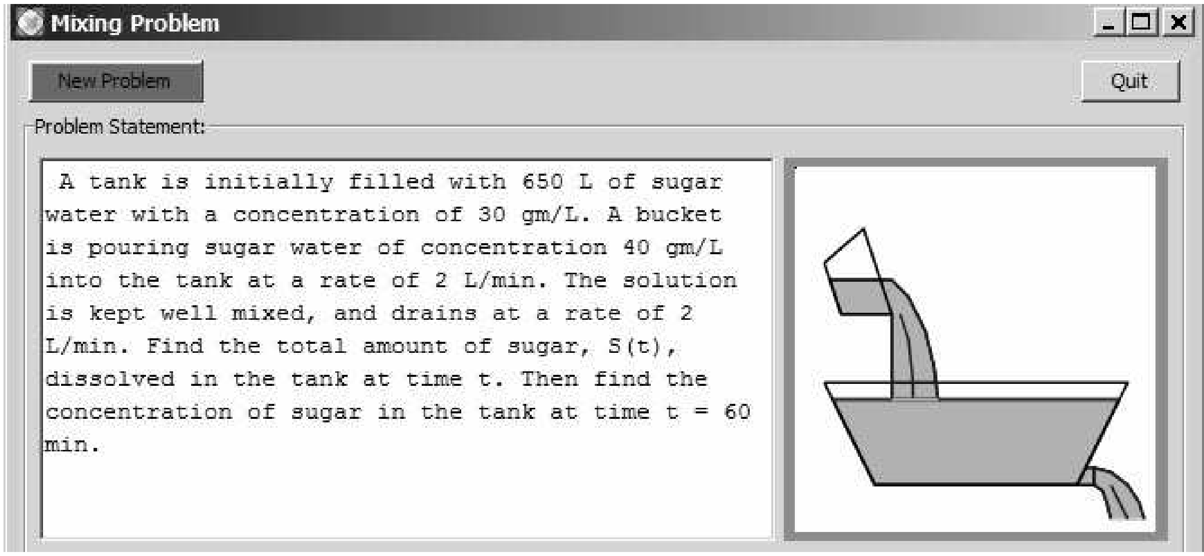
Solution: Separate: $\int \frac{dy}{y^2} = \int -x^3\,dx$ Integrate: $-\frac{1}{y} = -\frac{x^4}{4} + C$

Plug in the initial condition, $x = 1$ when $y = 2$: $-\frac{1}{2} = -\frac{1}{4} + C \quad \Rightarrow \quad C = -\frac{1}{4}$

Substitute for C and solve for y: $-\frac{1}{y} = -\frac{x^4}{4} - \frac{1}{4} \qquad \frac{1}{y} = \frac{x^4 + 1}{4} \qquad y = \frac{4}{x^4 + 1}$

Work Out (5 questions, Points indicated. Show all you work.)

14. (20 points)



a. (8 pts) Write the differential equation and initial condition for $S(t)$.

Solution: $\frac{dS}{dt}\frac{\text{gm}}{\text{min}} = 40\frac{\text{gm}}{\text{L}}\cdot 2\frac{\text{L}}{\text{min}} - \frac{S\,\text{gm}}{650\,\text{L}}\cdot 2\frac{\text{L}}{\text{min}}$ or $\frac{dS}{dt} = 80 - \frac{1}{325}S$

$S(0) = 30\,\frac{\text{gm}}{\text{L}}\,650\text{ L} = 19500\text{ gm}$

b. (9 pts) Solve the initial value problem for $S(t)$.

Solution Method 1: The equation is separable. Separate and integrate.

$\int \frac{dS}{80 - \frac{1}{325}S} = \int dt \qquad -325\ln\left|80 - \frac{1}{325}S\right| = t + C$

Solve: $\ln\left|80 - \frac{1}{325}S\right| = -\frac{t}{325} - \frac{C}{325} \qquad \left|80 - \frac{1}{325}S\right| = e^{-C/325}e^{-t/325}$

$80 - \frac{1}{325}S = Ae^{-t/325}$ where $A = \pm e^{-C/325}$ $\qquad S = 26000 - 325Ae^{-t/325}$

Use the initial condition: At $t = 0$, we have $S = 19500$. So

$19500 = 26000 - 325A, \qquad A = 20$

Substitute back: $S = 26000 - 6500e^{-t/325}$

Solution Method 2: The equation is linear: Put it in standard form:

$\frac{dS}{dt} + \frac{1}{325}S = 80$

Find the integrating factor: $I = \exp\left(\int \frac{1}{325}dt\right) = e^{t/325}$

Multiply the standard equation by I: $e^{t/325}\frac{dS}{dt} + \frac{1}{325}e^{t/325}S = 80e^{t/325}$

Recognize the left side as the derivative of a product: $\frac{d}{dt}(e^{t/325}S) = 80e^{t/325}$

Integrate and solve: $e^{t/325}S = \int 80e^{t/325}\,dt = 80\cdot 325\,e^{t/325} + C \qquad S = 26000 + Ce^{-t/325}$

Use the initial condition: At $t = 0$, we have $S = 19500$. So

$19500 = 26000 + C \qquad C = -6500$

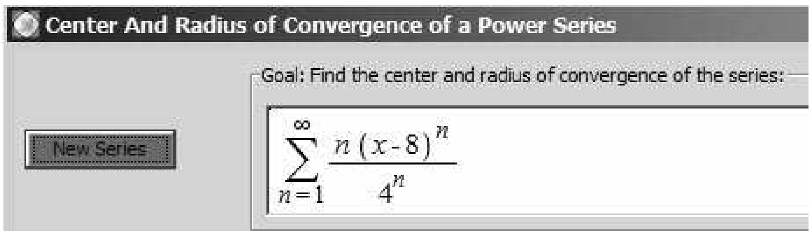
Substitute back: $S = 26000 - 6500e^{-t/325}$

c. (3 pts) Find the concentration in the tank at $t = 60$ min.

Solution: $S = 26000 - 6500e^{-60/325} = 26000 - 6500e^{-12/65}$

So the concentration is $\frac{S}{650} = \frac{26000 - 6500e^{-12/65}}{650} = 40 - 10e^{-12/65}$

15. $\left(20 \text{ points}\right)$



Also find the interval of convergence by checking the endpoints.

a. $\left(2 \text{ pts}\right)$ Identify the center: $a =$ ___8___

b. $\left(8 \text{ pts}\right)$ Find the radius of convergence:

Solution: Apply the Ratio Test:

$$\rho = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)|x-8|^{n+1}}{4^{n+1}} \frac{4^n}{n|x-8|^n} = \frac{|x-8|}{4} \lim_{n\to\infty} \frac{(n+1)}{n} = \frac{|x-8|}{4} < 1$$

$|x-8| < 4 \qquad 4 < x < 12$ $R =$ ___4___

c. $\left(8 \text{ pts}\right)$ Check the endpoints:

Solution:

$x = 4$: $\sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n} = \sum_{n=1}^{\infty} (-1)^n n \qquad \lim_{n\to\infty} (-1)^n n =$ divergent $\neq 0$

Diverges by the n^{th}-Term Divergence Test

$x = 12$: $\sum_{n=1}^{\infty} \frac{n(4)^n}{4^n} = \sum_{n=1}^{\infty} n \qquad \lim_{n\to\infty} n = \infty \neq 0$

Diverges by the n^{th}-Term Divergence Test

d. $\left(2 \text{ pts}\right)$ Summarize the interval of convergence: $I =$ ___$(4, 12)$___

16. (5 points) Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is absolutely convergent, convergent but not absolutely or divergent. Explain all tests you use.

Solution: $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent by the Alternating Series Test since the $(-1)^n$ says it is alternating, $\frac{1}{n^{1/3}}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n^{1/3}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ which is divergent because it is a p-series with $p = \frac{1}{3} < 1$.

So $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{1/3}}$ is convergent but not absolutely.

17. (5 points) The series $S = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges by the Integral Test.

If it is approximated by its 100^{th} partial sum S_{100}, compute the integral bound on the error in this approximation.

Solution: The bound is

$$|E_7| = |S - S_{100}| < \int_{100}^{\infty} \frac{1}{n^2+1}\,dn = \Big[\arctan(n)\Big]_{100}^{\infty} = \frac{\pi}{2} - \arctan(100) \qquad (\approx 0.01)$$

18. (5 points) Compute the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}}$.

Solution: $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ So $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!3^{2n+1}} = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$