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MATH 152H Exam 1 Spring 2016

Sections 201/202 (circle one) Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Compute $\int_0^1 x^3 e^{(x^2+1)}\,dx$

a. $\dfrac{e}{2}$ correct choice

b. $\dfrac{1}{2}(e^2-1)$

c. $\dfrac{e^2}{2}$

d. $\dfrac{1}{2}(e-1)$

e. $\dfrac{1}{2}(1-e)$

Solution: Use the substitution $w = x^2+1 \quad dw = 2x\,dx \quad \dfrac{1}{2}dw = x\,dx \quad x^2 = w-1$

$$\int_0^1 x^3 e^{(x^2+1)}\,dx = \frac{1}{2}\int_1^2 (w-1)e^w\,dw$$

Parts: $u = w-1 \quad dv = e^w\,dw$

$du = dw \quad v = e^w$

$$= \frac{1}{2}\left[(w-1)e^w - \int e^w\,dw\right] = \frac{1}{2}[(w-1)e^w - e^w] = \left[\frac{1}{2}(w-2)e^w\right]_1^2 = 0 - \left(\frac{1}{2}(-1)e^1\right) = \frac{e}{2}$$

2. Compute $\int_0^{\pi/2} x\cos x\,dx$

a. $\pi-1$

b. $1-\pi$

c. $\dfrac{\pi}{2}$

d. $1+\dfrac{\pi}{2}$

e. $\dfrac{\pi}{2}-1$ correct choice

Solution: Use integration by parts with $u = x \quad dv = \cos x\,dx$

$du = dx \quad v = \sin x$

$$\int_0^{\pi/2} x\cos x\,dx = \left[x\sin x - \int \sin x\,dx\right]_0^{\pi/2} = \left[x\sin x + \cos x\right]_0^{\pi/2}$$

$$= \left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (\cos 0) = \frac{\pi}{2} - 1$$

Check: $\dfrac{d}{dx}(x\sin x + \cos x) = \sin x + x\cos x - \sin x = x\cos x$

3. Compute $\int_1^2 x^2 \ln x\, dx$

a. $\frac{8}{3}\ln 2 - 1$

b. $\frac{8}{3}\ln 2 - \frac{7}{9}$ correct choice

c. $4\ln 2 - 1$

d. $4\ln 2 - 2$

e. $4\ln 2 - 3$

Solution: Use integration by parts with $u = \ln x \quad dv = x^2\,dx$, $du = \frac{1}{x}\,dx \quad v = \frac{x^3}{3}$

$$\int x^2 \ln x\, dx = \frac{x^3}{3}\ln x - \int \frac{x^3}{3}\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \int \frac{x^2}{3}\,dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$$

$$\int_1^2 x^2 \ln x\, dx = \left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_1^2 = \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(-\frac{1}{9}\right) = \frac{8}{3}\ln 2 - \frac{7}{9}$$

Check: $\frac{d}{dx}\left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right) = x^2\ln x + \frac{x^3}{3}\frac{1}{x} - \frac{x^2}{3} = x^2 \ln x$

4. Compute $\int_0^\pi \sin^3\theta\, d\theta$

a. $\frac{1}{3}$

b. $\frac{2}{3}$

c. 1

d. $\frac{4}{3}$ correct choice

e. 2

Solution: $u = \cos\theta \qquad du = -\sin\theta\, d\theta$

$$\int \sin^3\theta\, d\theta = \int (1-\cos^2\theta)\sin\theta\, d\theta = -\int (1-u^2)\,du = -\left[u - \frac{u^3}{3}\right] + C = -\cos\theta + \frac{\cos^3\theta}{3} + C$$

$$\int_0^{\pi/3} \sin^3\theta\, d\theta = \left[-\cos\theta + \frac{\cos^3\theta}{3}\right]_0^\pi = \left(-\cos\pi + \frac{1}{3}\cos^3\pi\right) - \left(-\cos 0 + \frac{1}{3}\cos^3 0\right)$$

$$= \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$$

Check: $\frac{d}{d\theta}\left(-\cos\theta + \frac{\cos^3\theta}{3}\right) = \sin\theta + \frac{1}{3}3\cos^2\theta(-\sin\theta) = \sin\theta(1-\cos^2\theta) = \sin^3\theta$

5. Compute $\int_0^{\pi/3} \sec^3\theta \tan^3\theta \, d\theta$

a. 0

b. $\frac{56}{15}$

c. $\frac{58}{15}$ correct choice

d. $\frac{128}{15}$

e. $\frac{136}{15}$

Solution: $u = \sec\theta \qquad du = \sec\theta\tan\theta\, d\theta \qquad \tan^2\theta = \sec^2\theta - 1 = u^2 - 1$

$$\int_0^{\pi/3} \sec^3\theta \tan^3\theta \, d\theta = \int_1^2 u^2(u^2-1)\, du = \int_1^2 (u^4 - u^2)\, du = \left[\frac{u^5}{5} - \frac{u^3}{3} \right]_1^2$$

$$= \left(\frac{2^5}{5} - \frac{2^3}{3} \right) - \left(\frac{1}{5} - \frac{1}{3} \right) = \frac{58}{15}$$

6. Compute $\int_0^{\pi/4} \cos 4\theta \cos 2\theta \, d\theta$

a. 1

b. $\frac{1}{2}$

c. $\frac{1}{3}$

d. $\frac{1}{6}$ correct choice

e. $\frac{1}{12}$

Solution: Add the identities:

$$\cos 6\theta = \cos(4\theta + 2\theta) = \cos 4\theta \cos 2\theta - \sin 4\theta \sin 2\theta$$

$$\cos 2\theta = \cos(4\theta - 2\theta) = \cos 4\theta \cos 2\theta + \sin 4\theta \sin 2\theta$$

$$\cos 6\theta + \cos 2\theta = 2\cos 4\theta \cos 2\theta$$

$$\int_0^{\pi/4} \cos 4\theta \cos 2\theta \, d\theta = \frac{1}{2} \int_0^{\pi/4} \cos 6\theta + \cos 2\theta \, d\theta = \frac{1}{2} \left[\frac{\sin 6\theta}{6} + \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{1}{6} \sin\frac{3\pi}{2} + \frac{1}{2}\sin\frac{\pi}{2} \right) = \frac{1}{2}\left(\frac{1}{6}(-1) + \frac{1}{2}(1) \right) = \frac{1}{6}$$

7. Find the area between, $y = x^2$ and $y = 3x$.

a. $\frac{81}{2}$

b. $\frac{27}{2}$

c. $\frac{9}{2}$ correct choice

d. $\frac{1}{2}$

e. $\frac{10}{3}$

Solution: The curves intersect when $x^2 = 3x$ or $x = 0, 3$.

$$A = \int_0^3 (3x - x^2)\, dx = \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - 9 = \frac{9}{2}$$

8. Find the average value of $f(x) = e^{3x}$ on the interval $[0, 2]$.

a. $\frac{1}{6}e^6$

b. $\frac{1}{6}(e^6 - 1)$ correct choice

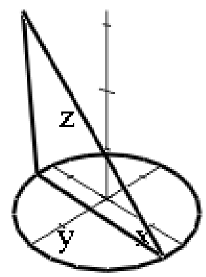
c. $\frac{1}{3}e^6$

d. $\frac{1}{2}(1 + e^6)$

e. $(e^6 - 1)$

Solution: $f_{\text{ave}} = \frac{1}{2}\int_0^2 e^{3x}\,dx = \left[\frac{1}{6}e^{3x}\right]_0^2 = \frac{1}{6}(e^6 - 1)$

9. A solid has a base which is a circle of radius 2 and has cross sections perpendicular to the y-axis which are isosceles right triangles with a leg on the base. Find the volume of the solid.



a. $\frac{64}{3}$ correct choice

b. $\frac{128}{3}$

c. $\frac{16}{3}\pi$

d. $\frac{32}{3}\pi$

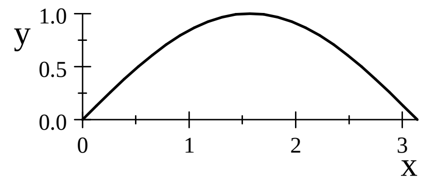
e. $\frac{32}{3}$

Solution: The width of the base is $W(y) = 2x = 2\sqrt{4 - y^2}$. The height is $H = W$

The area of the cross section is $A(y) = \frac{1}{2}WH = \frac{1}{2}2\sqrt{4 - y^2}\,2\sqrt{4 - y^2} = 2(4 - y^2)$.

$$V = \int_{-2}^{2} A(y)\,dy = \int_{-2}^{2} 2(4 - y^2)\,dy = 2\left[4y - \frac{y^3}{3}\right]_{-2}^{2} = 4\left(8 - \frac{8}{3}\right) = 32\left(\frac{2}{3}\right) = \frac{64}{3}$$

10. The region shown at the right is bounded by $y = \sin x$ and the x-axis. It is rotated about the line $y = -1$. Find the volume swept out.



a. $\frac{17\pi}{4}$
b. $\frac{9\pi}{2}$
c. $\frac{\pi^2}{2}$
d. $4\pi + 2\pi^2$
e. $4\pi + \frac{\pi^2}{2}$ correct choice

Solution: As an x-integral, a slice is vertical and rotates into a washer.

$$V = \int_0^\pi \pi R^2 - \pi r^2\,dx = \int_0^\pi \pi(1+\sin x)^2 - \pi(1)^2\,dx = \pi\int_0^\pi (2\sin x + \sin^2 x)\,dx$$

$$= \pi\int_0^\pi 2\sin x + \frac{1-\cos(2x)}{2}\,dx = \pi\left[-2\cos x + \frac{x}{2} - \frac{\sin(2x)}{4}\right]_0^\pi$$

$$= \pi\left(-2\cos\pi + \frac{\pi}{2}\right) - \pi(-2\cos 0) = 4\pi + \frac{1}{2}\pi^2$$

11. The region in Problem 10 is rotated about the the line $x = -1$. Which formula gives the volume swept out?

a. $\int_0^\pi \pi(1+\sin x)^2\,dx$
b. $\int_{-1}^\pi 2\pi x\sin x\,dx$
c. $\int_0^\pi \pi(x-1)\sin x\,dx$
d. $\int_0^\pi 2\pi(x+1)\sin x\,dx$ correct choice
e. $\int_0^\pi \pi\left((1+\sin x)^2 - 1\right)dx$

Solution: As an x-integral, a slice is vertical and rotates into a cylinder.

$$V = \int_0^\pi 2\pi rh\,dx = \int_0^\pi 2\pi(x+1)\sin x\,dx$$

This integral could be computed using integration by parts.

12. A certain spring is at rest when its mass is at $x = 3$.
It requires 24 Joules of work to stretch it from $x = 3$ to $x = 7$ meters.
What is the force required to maintain the mass at 7 meters?

a. 6 Newtons

b. 12 Newtons correct choice

c. 18 Newtons

d. 24 Newtons

e. 48 Newtons

Solution: The force is $F = k(x-3)$. The work is

$$W = \int_3^7 F\,dx = \int_3^7 k(x-3)\,dx = \left[\frac{1}{2}k(x-3)^2\right]_3^7 = \frac{1}{2}k(16-0) = 8k = 24.$$

So the force constant is $k = 3$. And the force is $F = kx = 3 \cdot 4 = 12$.

13. A 80 kg cable which is 20 meters long hangs down from the top of a building. A 5 kg bag of sand is at the bottom of the cable. How much work is done to lift the sand and the cable to the top of the building? Give your answer as a multiple of the acceleration of gravity g.

a. $800g$

b. $900g$ correct choice

c. $1600g$

d. $1700g$

e. $2200g$

Solution: The work to lift the bag of sand alone is $W = FD = (5g)(20) = 100g$.

The piece of rope at a distance y from the top of length dy

weighs $dF = \dfrac{80 \text{ kg}}{20 \text{ m}} g\,dy = 4g\,dy$ and is lifted a distance $D = y$.

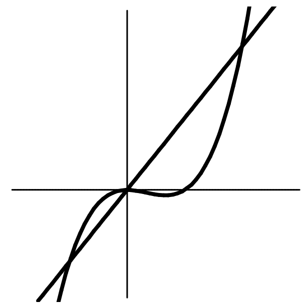
So the work is $W = \int_0^{20} D\,dF = \int_0^{20} y\,4g\,dy = \left[2y^2 g\right]_0^{20} = 800g$.

The total work is $W = 100g + 800g = 900g$.

Work Out: (Points indicated. Part credit possible. Show all work.)

14. $\left(15\text{ points}\right)$ Find the area between the cubic $y = x^3 - x^2$ and the line $y = 2x$.

Solution: The curves intersect when $x^3 - x^2 = 2x$
or $x^3 - x^2 - 2x = 0$ or $x(x^2 - x - 2) = 0$
or $x(x+1)(x-2) = 0$ or $x = -1, 0, 2$
Between -1 and 0, the cubic is above the line.
(Plug in $x = -1/2$.)
Between 0 and 2, the line is above the cubic.
(Plug in $x = 1$.)

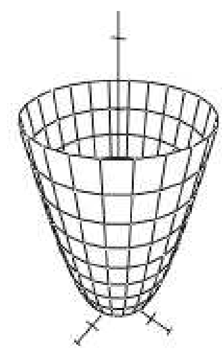


So the area is

$$A = \int_{-1}^{0} (x^3 - x^2 - 2x)\,dx + \int_{0}^{2} (2x - x^3 + x^2)\,dx = \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2\right]_{-1}^{0} + \left[x^2 - \frac{x^4}{4} + \frac{x^3}{3}\right]_{0}^{2}$$

$$= (0) - \left(\frac{1}{4} - \frac{-1}{3} - 1\right) + \left(4 - 4 + \frac{8}{3}\right) - (0) = \frac{-3 - 4 + 12 + 32}{12} = \frac{37}{12}$$

15. $\left(20\text{ points}\right)$ A water tower is made by rotating the curve $y = x^4$ about the y-axis, where x and y are in meters. If the tower is filled with water up to height $y = 25$ m, how much work is done to pump all the water out a faucet at height 30 m?
Assume the acceleration of gravity is $g = 9.8$ m/sec^2 and the density of water is $\rho = 1000$ kg/m^3.
You may give your answer as a multiple of $\rho g \pi$.



Solution: The cross section at height y is a circle of radius $x = y^{1/4}$
and hence area $A = \pi x^2 = \pi y^{1/2}$.
A slice of thickness dy has volume $dV = A\,dy = \pi y^{1/2}\,dy$ and mass $dm = \rho\,dV = \rho\pi y^{1/2}\,dy$.
This slice must be lifted a distance $h = 30 - y$.
So the work to lift it is $dW = dm\,g\,h = \rho\pi g(30 - y)y^{1/2}\,dy$.
The total work is

$$W = \int_0^{25} \rho\pi g(30 - y)y^{1/2}\,dy = \pi\rho g \int_0^{25} (30y^{1/2} - y^{3/2})\,dy = \pi\rho g\left[30\frac{2y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^{25}$$

$$= \pi\rho g\left(30\frac{2 \cdot 5^3}{3} - \frac{2 \cdot 5^5}{5}\right) = \pi\rho g(4 \cdot 5^4 - 2 \cdot 5^4) = 2 \cdot 5^4 \pi\rho g = 1250\pi\rho g$$

$$= 3.8485 \times 10^7 \text{ Joules}$$

16. (15 points) Consider the curve $y = f(x) = x^2$.

a. Find the tangent line at the generic point $x = p$. Call it $y = f_{\text{tan}}(x)$.

Solution: $f'(x) = 2x$.
$y = f_{\text{tan}}(x) = f(p) + f'(p)(x-p) = p^2 + 2p(x-p) = 2px - p^2$

b. The region between $y = f(x)$ and $y = f_{\text{tan}}(x)$ for $0 \le x \le 1$ is rotated about the y-axis. Find the volume swept out. Call it $V(p)$.

Solution: We use an x-integral. There are vertical slices which rotate into cylinders.

Radius $r = x$. Height $h = f(x) - f_{\text{tan}}(x) = x^2 - 2px + p^2$.

$$V(p) = \int_0^1 2\pi rh\,dx = 2\pi\int_0^1 x(x^2 - 2px + p^2)\,dx = 2\pi\int_0^1 (x^3 - 2px^2 + p^2x)\,dx$$
$$= 2\pi\left[\frac{x^4}{4} - 2p\frac{x^3}{3} + p^2\frac{x^2}{2}\right]_0^1 = 2\pi\left(\frac{1}{2}p^2 - \frac{2}{3}p + \frac{1}{4}\right)$$

c. Find the value of p which minimizes the volume $V(p)$.

Solution: $V(p) = 2\pi\left(\frac{1}{2}p^2 - \frac{2}{3}p + \frac{1}{4}\right)$
$V'(p) = 2\pi\left(p - \frac{2}{3}\right) = 0 \qquad \Rightarrow \qquad p = \frac{2}{3}$

d. Repeat steps (a), (b) and (c) for the general concave up curve $y = f(x) = g(x)$.

Note: You can't do the integral $\int_0^1 g(x)x\,dx$. So leave it unevaluated.

(a) **Solution**: $f'(x) = g'(x)$.
$y = f_{\text{tan}}(x) = f(p) + f'(p)(x-p) = g(p) + g'(p)(x-p)$

(b) **Solution**: Radius $r = x$. Height $h = f(x) - f_{\text{tan}}(x) = g(x) - g(p) - g'(p)(x-p)$.

$$V(p) = \int_0^1 2\pi rh\,dx = 2\pi\int_0^1 x(g(x) - g(p) - g'(p)(x-p))\,dx$$
$$= 2\pi\int_0^1 (g(x)x - g(p)x - g'(p)(x^2 - px))\,dx$$
$$= 2\pi\int_0^1 g(x)x\,dx + 2\pi\left[-g(p)\frac{x^2}{2} - g'(p)\left(\frac{x^3}{3} - p\frac{x^2}{2}\right)\right]_0^1$$
$$= 2\pi\left[\int_0^1 g(x)x\,dx - \frac{1}{2}g(p) - g'(p)\left(\frac{1}{3} - \frac{1}{2}p\right)\right]$$

(c) $V'(p) = 2\pi\left[-\frac{1}{2}g'(p) - g''(p)\left(\frac{1}{3} - \frac{1}{2}p\right) - g'(p)\left(-\frac{1}{2}\right)\right]$
$= 2\pi\left[g''(p)\left(\frac{1}{2}p - \frac{1}{3}\right)\right] = 0 \qquad \Rightarrow \qquad p = \frac{2}{3}$
provided $g''(p) \neq 0$ which is true because $g(x)$ is everywhere concave up.
$2\pi\left(p - \frac{2}{3}\right) = 0 \qquad p = \frac{2}{3}$

e. What do you conclude?

Solution: For any concave up curve $y = f(x)$, if the region between the curve and its tangent line $y = f_{\text{tan}}(x)$ at a point p is rotated about the y-axis, then the volume swept out is a minimum when $p = \frac{2}{3}$.