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MATH 152H Exam 2 Spring 2016

Sections 201/202 (circle one) Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the general partial fraction expansion of $f(x) = \dfrac{x^2-4}{(x^3+4x)(x^4-16)}$.

a. $\dfrac{A}{x} + \dfrac{Dx+E}{(x^2+4)^2}$

b. $\dfrac{A}{x} + \dfrac{Bx+C}{x^2-4} + \dfrac{Dx+E}{(x^2+4)^2}$

c. $\dfrac{A}{x} + \dfrac{Bx+C}{x^2-4} + \dfrac{Dx+E}{x^2+4}$

d. $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2}$ correct choice

e. $\dfrac{A}{x} + \dfrac{Bx}{x^2-4} + \dfrac{Dx}{(x^2+4)^2}$

Solution: $f(x) = \dfrac{x^2-4}{x(x^2+4)(x^2-4)(x^2+4)} = \dfrac{1}{x(x^2+4)^2}$

$f(x) = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+4} + \dfrac{Dx+E}{(x^2+4)^2}$

2. In the partial fraction expansion $\dfrac{x^3-4x-32}{x^4-16} = \dfrac{Ax+B}{x^2+4} + \dfrac{C}{x+2} + \dfrac{D}{x-2}$, which coefficient is INCORRECT?

a. $A = 1$

b. $B = 4$

c. $C = 1$

d. $D = 1$ correct choice

e. All of the above are correct.

Solution: Clear the denominator:

$$x^3-4x-32 = (Ax+B)(x^2-4) + C(x^2+4)(x-2) + D(x^2+4)(x+2)$$

Plug in $x=-2$: $-8+8-32 = (-A2+B)(0) + C(4+4)(-2-2) + D(4+4)(0)$

$-32 = C(-32) \qquad C = 1$

Plug in $x=2$: $8-8-32 = (A2+B)(0) + C(4+4)(0) + D(4+4)(2+2)$

$-32 = D(32) \qquad D = -1 \neq 1$ $\Leftarrow\Leftarrow\Leftarrow$ incorrect

Plug in $x=0$: $-32 = B(-4) + (4)(-2) - 1(4)(2) \qquad -32 = -4B - 16$

$4B = 16 \qquad B = 4$

Coeff of x^3 : $1 = A + C + D = A + 1 - 1 = A$

3. Compute $\int_{-1}^{4} \frac{1}{x^2}\,dx$

a. $\frac{5}{4}$

b. $-\frac{5}{4}$

c. diverges to $+\infty$ correct choice

d. diverges to $-\infty$

e. diverges but not to $\pm\infty$

Solution: $\int_{-1}^{4} \frac{1}{x^2}\,dx = \int_{-1}^{0} x^{-2}\,dx + \int_{0}^{4} x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_{-1}^{0} + \left[\frac{x^{-1}}{-1}\right]_{0}^{4}$

$$= \left[-\frac{1}{x}\right]_{-1}^{0} + \left[-\frac{1}{x}\right]_{0}^{4} = \lim_{x\to 0^-}\left(-\frac{1}{x}\right) - \left(-\frac{1}{-1}\right) + \left(-\frac{1}{4}\right) - \lim_{x\to 0^+}\left(-\frac{1}{x}\right)$$

$$= \text{“}\left(-\frac{1}{0^-}\right)\text{”} - 1 - \frac{1}{4} - \text{“}\left(-\frac{1}{0^+}\right)\text{”} = \infty - \frac{5}{4} + \infty = \infty$$

4. The integral $\int_{\pi}^{\infty} \frac{3+\sin(x)}{x}\,dx$ is

a. divergent by comparison with $\int_{\pi}^{\infty} \frac{2}{x}\,dx$. correct choice

b. divergent by comparison with $\int_{\pi}^{\infty} \frac{3}{x}\,dx$.

c. divergent by comparison with $\int_{\pi}^{\infty} \frac{4}{x}\,dx$.

d. convergent by comparison with $\int_{\pi}^{\infty} \frac{2}{x}\,dx$.

e. convergent by comparison with $\int_{\pi}^{\infty} \frac{4}{x}\,dx$.

Solution: The integrals $\int_{\pi}^{\infty} \frac{4}{x}\,dx$ and $\int_{\pi}^{\infty} \frac{2}{x}\,dx$ both diverge to $+\infty$ because

$$\int_{\pi}^{\infty} \frac{k}{x}\,dx = k\left[\ln x\right]_{\pi}^{\infty} = \infty$$

Since $\frac{3+\sin(x)}{x} \ge \frac{2}{x}$, we have $\int_{\pi}^{\infty} \frac{3+\sin(x)}{x}\,dx \ge \infty$ and so diverges.

Since $\frac{3+\sin(x)}{x} \le \frac{4}{x}$, we have $\int_{\pi}^{\infty} \frac{3+\sin(x)}{x}\,dx \le \infty$ and cannot conclude anything.

5. Compute $\int_0^{4/3} \frac{1}{9x^2+16}\,dx$

a. $\frac{16}{9}$

b. $\frac{1}{9}$

c. $\frac{\pi}{12}$

d. $\frac{\pi}{24}$

e. $\frac{\pi}{48}$ correct choice

Solution: $3x = 4\tan\theta \qquad x = \frac{4}{3}\tan\theta \qquad dx = \frac{4}{3}\sec^2\theta\,d\theta \qquad \tan^2\theta + 1 = \sec^2\theta$

$$\int_0^{4/3} \frac{1}{9x^2+16}\,dx = \int_{x=0}^{4/3} \frac{1}{16\tan^2\theta+16}\,\frac{4}{3}\sec^2\theta\,d\theta = \frac{1}{12}\int_{x=0}^{4/3} 1\,d\theta = \left[\frac{\theta}{12}\right]_{x=0}^{4/3}$$

$$= \left[\frac{1}{12}\arctan\frac{3x}{4}\right]_0^{4/3} = \frac{1}{12}(\arctan 1 - \arctan 0) = \frac{\pi}{48}$$

6. Which of the following integrals results after performing an appropriate trig substitution for

$$\int_1^{4/3} \frac{1}{25x^2-16}\,dx$$

a. $\frac{1}{20}\int_{\mathrm{arcsec}(5/3)}^{\mathrm{arcsec}(5/4)} \csc\theta\,d\theta$

b. $\frac{1}{20}\int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \csc\theta\,d\theta$ correct choice

c. $\frac{5}{64}\int_{\mathrm{arcsec}(5/3)}^{\mathrm{arcsec}(5/4)} \sec\theta\,d\theta$

d. $\frac{5}{64}\int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \sec\theta\,d\theta$

e. $\frac{1}{16}\int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \cot^2\theta\,d\theta$

Solution: $5x = 4\sec\theta$ because from the limits $x > 4/5$.

So $\quad x = \frac{4}{5}\sec\theta \qquad dx = \frac{4}{5}\sec\theta\tan\theta\,d\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

$x = 1$ when $\theta = \mathrm{arcsec}\,\frac{5}{4} \qquad x = \frac{4}{3}$ when $\theta = \mathrm{arcsec}\,\frac{5}{3}$

$$\int_1^{4/3} \frac{1}{25x^2-16}\,dx = \int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \frac{1}{16\sec^2\theta-16}\,\frac{4}{5}\sec\theta\tan\theta\,d\theta = \frac{1}{20}\int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \frac{\sec\theta}{\tan\theta}\,d\theta$$

$$= \frac{1}{20}\int_{\mathrm{arcsec}(5/4)}^{\mathrm{arcsec}(5/3)} \csc\theta\,d\theta$$

7. Compute $\int_0^{3/5} \frac{1}{25x^2 - 16}\, dx$

a. $-\frac{1}{20}$

b. $\frac{1}{20}$

c. $-\frac{1}{40}\ln 7$ correct choice

d. $\frac{1}{20} - \frac{1}{40}\ln 7$

e. $\frac{1}{40}\ln 7 - \frac{1}{20}$

Solution: $5x = 4\sin\theta$ because from the limits $x < 4/5$.

So $x = \frac{4}{5}\sin\theta \qquad dx = \frac{4}{5}\cos\theta\, d\theta \qquad \sin^2\theta - 1 = -\cos^2\theta$

$$\int_0^{3/5} \frac{1}{25x^2 - 16}\, dx = \int_{x=0}^{3/5} \frac{1}{16\sin^2\theta - 16}\, \frac{4}{5}\cos\theta\, d\theta = \frac{1}{20}\int_{x=0}^{3/5} \frac{\cos\theta}{-\cos^2\theta}\, d\theta$$

$$= -\frac{1}{20}\int_{x=0}^{3/5} \sec\theta\, d\theta = \left[-\frac{1}{20}\ln|\sec\theta + \tan\theta|\right]_{x=0}^{3/5}$$

Draw a triangle with $\sin\theta = \frac{5x}{4}$: $Opp = 5x$, $Hyp = 4$, $Adj = \sqrt{16 - 25x^2}$.

Then $\sec\theta = \frac{4}{\sqrt{16 - 25x^2}} \qquad \tan\theta = \frac{5x}{\sqrt{16 - 25x^2}}$

$$\int_0^{3/5} \frac{1}{25x^2 - 16}\, dx = \left[-\frac{1}{20}\ln\left|\frac{4 + 5x}{\sqrt{16 - 25x^2}}\right|\right]_{x=0}^{3/5} = -\frac{1}{20}\left(\ln\left|\frac{4+3}{\sqrt{16-9}}\right| - \ln\left|\frac{4}{\sqrt{16}}\right|\right)$$

$$= -\frac{1}{20}\left(\ln\frac{7}{\sqrt{7}} - \ln 1\right) = -\frac{1}{40}\ln 7$$

The integral can also be done by partial fractions.

8. Find the length of the curve $x = 2t^2$, and $y = t^3$ for $0 \le t \le 1$.

a. 732

b. 108

c. 7

d. $\frac{61}{27}$ correct choice

e. $\frac{1}{3}$

Solution: As a t-integral, the differential of arc length is

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \sqrt{(4t)^2 + (3t^2)^2}\, dt = \sqrt{16t^2 + 9t^4}\, dt = t\sqrt{16 + 9t^2}\, dt$$

So the arc length is

$$L = \int ds = \int_0^1 t\sqrt{16 + 9t^2}\, dt \qquad u = 16 + 9t^2 \qquad du = 18t\, dt$$

$$L = \frac{1}{18}\int_{16}^{25} \sqrt{u}\, du = \left[\frac{1}{18}\frac{2u^{3/2}}{3}\right]_{16}^{25} = \frac{1}{27}(125 - 64) = \frac{61}{27}$$

9. The curve $y = \sin x$ for $0 \le x \le \pi$ is rotated about the x-axis.
Which integral gives the area of the resulting surface?

a. $\int_0^\pi 2\pi x\sqrt{1+\cos^2 x}\,dx$

b. $\int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}\,dx$ correct choice

c. $\int_0^\pi 2\pi \cos x\sqrt{1+\sin^2 x}\,dx$

d. $\int_0^\pi \pi \cos x\sqrt{1+\sin^2 x}\,dx$

e. $\int_0^\pi \pi \sin x\,dx$

Solution: As an x-integral, the differential of arc length is

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\cos^2 x}\,dx$$

The radius of revolution is $r = y = \sin x$. So the surface area is

$$A = \int_0^\pi 2\pi r\,ds = \int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}\,dx$$

10. The curve $y = \sin x$ for $0 \le x \le \pi$ is rotated about the y-axis.
Which integral gives the area of the resulting surface?

a. $\int_0^\pi 2\pi x\sqrt{1+\cos^2 x}\,dx$ correct choice

b. $\int_0^\pi 2\pi \sin x\sqrt{1+\cos^2 x}\,dx$

c. $\int_0^\pi 2\pi \cos x\sqrt{1+\sin^2 x}\,dx$

d. $\int_0^\pi \pi \cos x\sqrt{1+\sin^2 x}\,dx$

e. $\int_0^\pi \pi \sin x\,dx$

Solution: As an x-integral, the differential of arc length is

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\cos^2 x}\,dx$$

The radius of revolution is $r = x$. So the surface area is

$$A = \int_0^\pi 2\pi r\,ds = \int_0^\pi 2\pi x\sqrt{1+\cos^2 x}\,dx$$

11. Consider the sequence $a_n = \arctan\left(\frac{3n-6}{\sqrt{3}\,n-6}\right)$. Compute $\lim_{n\to\infty} a_n$.

- **a.** π
- **b.** $\frac{\pi}{2}$
- **c.** $\frac{\pi}{3}$ correct choice
- **d.** $\frac{\pi}{4}$
- **e.** $\frac{\pi}{6}$

Solution: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \arctan\left(\frac{3n-6}{\sqrt{3}\,n-6}\right) = \arctan\left(\lim_{n\to\infty} \frac{3n-6}{\sqrt{3}\,n-6}\right)$

$$= \arctan\left(\frac{3}{\sqrt{3}}\right) = \arctan\sqrt{3} = \frac{\pi}{3}$$

12. $\sum_{n=1}^{\infty} 3\frac{2^{2n}}{3^n} =$

- **a.** -12
- **b.** 12
- **c.** -9
- **d.** 9
- **e.** diverges correct choice

Solution: $a = 3\frac{2^2}{3} = 4$ $\quad r = \frac{2^2}{3} = \frac{4}{3} > 1$ diverges

13. $\sum_{n=1}^{\infty} 3\frac{2^n}{3^n} =$

- **a.** 3
- **b.** 6 correct choice
- **c.** 9
- **d.** 12
- **e.** diverges

Solution: $a = 3\frac{2}{3} = 2$ $\quad r = \frac{2}{3} < 1$

$$\sum_{n=1}^{\infty} 3\frac{2^n}{3^n} = \frac{a}{1-r} = \frac{2}{1-\frac{2}{3}} = 6$$

14. $\sum_{n=3}^{\infty} \ln\left(\frac{n^2}{n^2-1}\right) =$

a. $\ln\frac{3}{4}$

b. $\ln\frac{3}{2}$ correct choice

c. $\ln 3$

d. $\ln\frac{1}{4}$

e. $\ln\frac{1}{2}$

Solution: $\ln\left(\frac{n^2}{n^2-1}\right) = \ln\left(\frac{n}{(n-1)}\frac{n}{(n+1)}\right) = \ln\left(\frac{n}{n-1}\right) - \ln\left(\frac{n+1}{n}\right)$

$$S_k = \sum_{n=3}^{k}\left[\ln\left(\frac{n}{n-1}\right) - \ln\left(\frac{n+1}{n}\right)\right] = \left[\ln\frac{3}{2} - \ln\frac{4}{3}\right] + \left[\ln\frac{4}{3} - \ln\frac{5}{3}\right] + \cdots + \left[\ln\left(\frac{k}{k-1}\right) - \ln\left(\frac{k+1}{k}\right)\right]$$

$$= \ln\frac{3}{2} - \ln\left(\frac{k+1}{k}\right)$$

$$S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}\left[\ln\frac{3}{2} - \ln\left(\frac{k+1}{k}\right)\right] = \ln\frac{3}{2}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

15. (15 points) Compute $\int \frac{3(x-3)^2}{x^4-81}\,dx$. Simplify to a single $\ln$.

Solution: $\frac{3(x-3)^2}{x^4-81} = \frac{3(x-3)}{(x^2+9)(x+3)} = \frac{Ax+B}{x^2+9} + \frac{C}{x+3}$

Clear the denominator: $3(x-3) = (Ax+B)(x+3) + C(x^2+9)$

Plug in $x = -3$: $-18 = (Ax+B)(0) + C(18)$ $C = -1$

Plug in $x = 0$: $-9 = B(3) + C(9) = 3B - 9$ $B = 0$

Plug in $x = 3$: $0 = (A3+B)(6) + C(9+9) = 18A - 18$ $A = 1$

So $\frac{3(x-3)^2}{x^4-81} = \frac{x}{x^2+9} - \frac{1}{x+3}$

So the integral is

$$\int \frac{3(x-3)^2}{x^4-81}\,dx = \int \frac{x}{x^2+9} - \frac{1}{x+3}\,dx = \frac{1}{2}\ln|x^2+9| - \ln|x+3| + C = \ln\left|\frac{\sqrt{x^2+9}}{x+3}\right| + C$$

16. (15 points) Compute $\int_1^{4/3} \frac{1}{\sqrt{25x^2 - 16}}\, dx$. Simplify to a single $\ln$.

Solution: $5x = 4\sec\theta$ because from the square root $x > 4/5$.

So $x = \frac{4}{5}\sec\theta \qquad dx = \frac{4}{5}\sec\theta\tan\theta\, d\theta \qquad \sec^2\theta - 1 = \tan^2\theta$

$$\int_1^{4/3} \frac{1}{\sqrt{25x^2 - 16}}\, dx = \int_{x=1}^{4/3} \frac{1}{\sqrt{16\sec^2\theta - 16}}\, \frac{4}{5}\sec\theta\tan\theta\, d\theta = \frac{1}{5}\int_{x=1}^{4/3} \sec\theta\, d\theta$$

$$= \left[\frac{1}{5}\ln|\sec\theta + \tan\theta|\right]_{x=1}^{4/3}$$

Draw a triangle with $\sec\theta = \frac{5x}{4}$: $Hyp = 5x$, $Adj = 4$, $Opp = \sqrt{25x^2 - 16}$.

Then $\tan\theta = \frac{\sqrt{25x^2 - 16}}{4}$

$$\int_1^{4/3} \frac{1}{\sqrt{25x^2 - 16}}\, dx = \left[\frac{1}{5}\ln\left|\frac{5x}{4} + \frac{\sqrt{25x^2 - 16}}{4}\right|\right]_{x=1}^{4/3}$$

$$= \frac{1}{5}\left(\ln\left|\frac{5}{4}\cdot\frac{4}{3} + \frac{\sqrt{25\left(\frac{4}{3}\right)^2 - 16}}{4}\right| - \ln\left|\frac{5}{4} + \frac{3}{4}\right|\right)$$

$$= \frac{1}{5}\left(\ln\left|\frac{5}{3} + \frac{4}{3}\right| - \ln 2\right) = \frac{1}{5}\ln\frac{3}{2}$$

17. (18 points) Consider the sequence defined recurrsively by

$$a_1 = 2 \qquad \text{and} \qquad a_{n+1} = 5 - \frac{4}{a_n}$$

a. (6 pts) Use mathematical induction to show a_n is positive and increasing for all n, i.e. $a_{n+1} \geq a_n > 0$.

i. To get started, show $a_2 \geq a_1 > 0$:

Solution: We compute $a_2 = 5 - \frac{4}{a_1} = 5 - \frac{4}{2} = 3 \geq 2 = a_1 > 0$

ii. Assume $a_{k+1} \geq a_k > 0$ and prove $a_{k+2} \geq a_{k+1} > 0$:

Solution: $a_{k+1} \geq a_k \quad \Rightarrow \quad \frac{1}{a_{k+1}} \leq \frac{1}{a_k}$ since a_{k+1} and a_k are positive.

$\Rightarrow \quad -\frac{4}{a_{k+1}} \geq -\frac{4}{a_k}$ since multiplying by a negative reverses an inequality.

$\Rightarrow \quad 5 - \frac{4}{a_{k+1}} \geq 5 - \frac{4}{a_k}$ since adding 5 preserves the direction of the inequality.

but this says $a_{k+2} \geq a_{k+1}$ and we already know $a_{k+1} > 0$

Therefore, $a_{n+1} \geq a_n > 0$ for all n.

b. (6 pts) Use mathematical induction to show a_n is bounded above by 8.

i. To get started, show $a_1 < 8$ and $a_2 < 8$:

Solution: $a_1 = 2 < 8$ and $a_2 = 3 < 8$

ii. Assume $a_k < 8$ and prove $a_{k+1} < 8$:

Solution: $a_k < 8 \quad \Rightarrow \quad \frac{1}{a_k} > \frac{1}{8}$ since a_k is positive.

$\Rightarrow \quad -\frac{4}{a_k} < -\frac{4}{8} = -\frac{1}{2}$ since multiplying by a negative reverses an inequality.

$\Rightarrow \quad 5 - \frac{4}{a_k} < 5 - \frac{1}{2} = 4.5$ since adding 5 preserves the direction of the inequality.

but this says $a_{k+1} < 4.5 < 8$.

Therefore, $a_n < 8$ for all n.

c. (6 pts) What do you conclude about $\lim\limits_{n\to\infty} a_n$? Find $\lim\limits_{n\to\infty} a_n$.

Solution: Since a_n is increasing and bounded above, we conclude $\lim\limits_{n\to\infty} a_n$ exists.

Let $\lim\limits_{n\to\infty} a_n = L$. Then we also have $\lim\limits_{n\to\infty} a_{n+1} = L$.

We compute the limit of $a_{n+1} = 5 - \frac{4}{a_n}$:

$$\lim_{n\to\infty} a_{n+1} = 5 - \frac{4}{\lim\limits_{n\to\infty} a_n} \quad \Rightarrow \quad L = 5 - \frac{4}{L} \quad \Rightarrow \quad L^2 - 5L + 4 = 0$$

$$\Rightarrow \quad (L-1)(L-4) = 0 \quad \Rightarrow \quad L = 1, 4$$

Since a_n starts at $a_1 = 2$ and increases, and the limit must exist, we must have $L = 4$.