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14	/20
15	/16
16	/18
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MATH 152H Exam 3 Spring 2016

Sections 201/202 (circle one) Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. The series $S = \sum_{n=1}^{\infty}(3^{1/n} - 3^{1/(n+1)})$ is

a. absolutely convergent. correct choice

b. conditionally convergent.

c. divergent by the n^{th} Term Divergence Test

d. divergent by the Alternating Series Test.

e. divergent because it is the difference between two p-series with $p = \frac{1}{n} < 1$ and $p = \frac{1}{n+1} < 1$.

Solution: The series is telescoping and

$$S_k = \sum_{n=1}^{k}(3^{1/n} - 3^{1/(n+1))}) = (3^1 - 3^{1/2}) + (3^{1/2} - 3^{1/3}) + \cdots + (3^{1/k} - 3^{1/(k+1)}) = 3^1 - 3^{1/(k+1)}$$

So $S = \lim_{k\to\infty} S_k = \lim_{k\to\infty}(3^1 - 3^{1/(k+1)}) = 3^1 - 3^0 = 3 - 1 = 2$. So it is convergent.

The related absolute series is the same series since $3^{1/n} - 3^{1/(n+1)} > 0$.

2. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is

a. convergent and absolutely convergent.

b. absolutely convergent but not convergent.

c. convergent but not absolutely convergent. correct choice

d. divergent.

Solution: $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent because it is an alternating decreasing series and $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$.

The related absolute series is $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ which is divergent because it is a p-series with $p = \frac{1}{2} < 1$. So $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ is convergent but not absolutely convergent i.e. conditionally convergent.

3. If $f(x) = \sin(x^2)$, compute $f^{(6)}(0)$.

- **a**. $-6 \cdot 6!$
- **b**. $-6 \cdot 3!$
- **c**. $6 \cdot 3!$
- **d**. $-5!$ correct choice
- **e**. $5!$

Solution: $\sin t = t - \frac{1}{6}t^3 + \frac{1}{5!}t^5 + \cdots \qquad \sin x^2 = x^2 - \frac{1}{6}x^6 + \frac{1}{5!}x^{10} + \cdots$

The coefficient of x^6 is $-\frac{1}{6}$. The coefficient of x^6 in the general Maclaurin series is $\frac{f^{(6)}(0)}{6!}$.

Equating we have $\frac{f^{(6)}(0)}{6!} = -\frac{1}{6}$. So $f^{(6)}(0) = -\frac{6!}{6} = -5!$

4. Compute $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!}$ HINT: Think about a known Maclaurin series.

- **a**. $\frac{1}{e^2}$ correct choice
- **b**. $\frac{-1}{e^2}$
- **c**. $\frac{1}{2}$
- **d**. $-e^2$
- **e**. The series diverges.

Solution: $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ So $\sum_{n=0}^{\infty} \frac{(-2)^n}{n!} = e^{-2} = \frac{1}{e^2}$

5. Compute $\lim_{x \to 0} \frac{\cos(x^3) - 1 + \frac{1}{2}x^6}{x^{12}}$

- **a**. $-\frac{1}{24}$
- **b**. $-\frac{1}{4}$
- **c**. $\frac{1}{4}$
- **d**. $\frac{1}{6}$
- **e**. $\frac{1}{24}$ correct choice

Solution: $\lim_{x \to 0} \frac{\cos(x^3) - 1 + \frac{1}{2}x^6}{x^{12}} = \lim_{x \to 0} \frac{\left[1 - \frac{(x^3)^2}{2} + \frac{(x^3)^4}{4!} \cdots\right] - 1 + \frac{1}{2}x^6}{x^{12}}$

$= \lim_{x \to 0} \left[\frac{(x^3)^4}{24x^{12}} + \cdots\right] = \frac{1}{24}$

6. The series $S = \sum_{n=0}^{\infty} \frac{3^n}{1+4^n}$ satisfies

a. $S = 0$
b. $0 < S < 4$ correct choice
c. $S = 4$
d. $S > 4$
e. The series diverges.

Solution: $\sum_{n=0}^{\infty} \frac{3^n}{4^n}$ is geometric. $a = 1$ and $r = \frac{3}{4}$.

Since $|r| < 1$, $\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \frac{1}{1-\frac{3}{4}} = 4$ So $\sum_{n=0}^{\infty} \frac{3^n}{1+4^n} < \sum_{n=0}^{\infty} \frac{3^n}{4^n} = 4$

Also $\sum_{n=0}^{\infty} \frac{3^n}{1+4^n} > 0$ because all terms are positive.

7. Find the equation of the sphere whose diameter has endpoints at $(2,1,6)$ and $(4,-3,8)$.

a. $(x+6)^2 + (y-2)^2 + (z+14)^2 = 6$
b. $(x+3)^2 + (y-1)^2 + (z+7)^2 = 24$
c. $(x-3)^2 + (y+1)^2 + (z-7)^2 = 6$ correct choice
d. $(x-3)^2 + (y+1)^2 + (z-7)^2 = 24$
e. $(x-6)^2 + (y+2)^2 + (z-14)^2 = 24$

Solution: The midpoint is $\frac{1}{2}(2+4, 1-3, 6+8) = (3,-1,7)$.

The radius is the distance from $(2,1,6)$ to $(3,-1,7)$:

$r = d((2,1,6),(3,-1,7)) = \sqrt{(3-2)^2 + (-1-1)^2 + (7-6)^2} = \sqrt{1+4+1} = \sqrt{6}$

The circle is $(x-3)^2 + (y+1)^2 + (z-7)^2 = 6$.

8. A triangle has vertices $A = (1,2,3)$, $B = (2,3,3)$ and $C = (1,3,2)$. Find the angle at A.

a. $0°$
b. $30°$
c. $45°$
d. $60°$ correct choice
e. $90°$

Solution: $\overrightarrow{AB} = B - A = (2,3,3) - (1,2,3) = (1,1,0)$ $\overrightarrow{AC} = C - A = (1,3,2) - (1,2,3) = (0,1,-1)$

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 0 + 1 + 0 = 1$ $\left|\overrightarrow{AB}\right| = \sqrt{1+1+0} = \sqrt{2}$ $\left|\overrightarrow{AC}\right| = \sqrt{0+1+1} = \sqrt{2}$

$\cos\theta = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$ $\theta = 60°$

9. The 3rd degree Taylor polynomial for $f(x) = \frac{1}{\sqrt{x}}$ at $x = 4$ is

$$T_3(x) = f(4) + f'(4)(x-4) + \frac{1}{2}f''(4)(x-4)^2 + \frac{1}{6}f'''(4)(x-4)^3$$
$$= \frac{1}{2} - \frac{1}{16}(x-4) + \frac{3}{256}(x-4)^2 - \frac{5}{2048}(x-4)^3$$

and the third derivative is $f'''(x) = \frac{-15}{8}x^{-7/2}$.

If the 2nd degree Taylor polynomial $T_2(x)$ is used to approximate $f(6) = \frac{1}{\sqrt{6}}$ we get

$$T_2(6) = \frac{1}{2} - \frac{1}{16}(6-4) + \frac{3}{256}(6-4)^2 = \frac{27}{64} = .4219$$

Which of the following is the best bound on the error in this approximation?

a. $|E_2| < \frac{5}{128} \approx .0391$
b. $|E_2| < \frac{5}{256} \approx .0195$ correct choice
c. $|E_2| < \frac{15}{128} \approx .1172$
d. $|E_2| < \frac{15}{256} \approx .0586$
e. $|E_2| < \frac{15}{512} \approx .0293$

Solution Method 1: This is the beginning of an alternating series.
So the error in the approximation is less than the absolute value of the next term.

$$|E_2| < |a_3| = \left|\frac{5}{2048}(6-4)^3\right| = \frac{5}{256} \approx .0195.$$

Solution Methiod 2: The Taylor Remainder Theorem says:
If you use $T_2(x)$, to approximate the function, $f(x)$, then the remainder, $E_2(x) = f(x) - T_2(x)$, is bounded by $|E_2(x)| \le \frac{M(x-a)^3}{3!}$ where a is the center and $M \ge |f^{(3)}(c)|$ for all c between a and x.
With $x = 6$, this says:
$E_2(6)$ is bounded by $|E_2(6)| \le \frac{M(6-4)^3}{3!}$ where $M \ge |f^{(3)}(c)|$ for all c between 4 and 6.
The maximum of $|f^{(3)}(x)| = \frac{15}{8}x^{-7/2}$ on $[4,6]$ occurs at $x = 4$. So we take $M = \frac{15}{8\sqrt{4}^7} = \frac{15}{1024}$.

Then $|R_2(x)| \le \frac{15}{1024}\frac{(6-4)^3}{(3)!} = \frac{5}{256} \approx .0195.$

Both methods give the same answer.

10. Suppose the function $y = f(x)$ is the solution of $\frac{dy}{dx} = x^2 - y^2$ satisfying the initial condition $f(1) = 2$. Find $f'(1)$.

a. -1
b. -2
c. -3 correct choice
d. 1
e. 2

Solution: As a solution, $y = f(x)$ satisfies $\frac{df}{dx} = x^2 - f(x)^2$. So $f'(1) = 1^2 - f(1)^2 = 1 - 2^2 = -3$.

11. Find the solution, $y = f(x)$, of the differential equation $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}$ satisfying the initial condition $f(0) = 3$. What is $f(3)$?

a. $3\sqrt{2}$
b. $3\sqrt[3]{2}$ correct choice
c. $\sqrt[3]{36}$
d. 6
e. 18

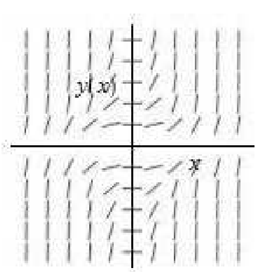
Solution: We separate: $y^2\,dy = x^2\,dx$ and integrate: $\dfrac{y^3}{3} = \dfrac{x^3}{3} + C$.
From the initial condition, when $x = 0$, $y = 3$. So $\dfrac{27}{3} = \dfrac{0^3}{3} + C$. So $C = 9$.
Then $y^3 = x^3 + 3C = x^3 + 27$ and $y = (x^3 + 27)^{1/3} = f(x)$. So $f(3) = 54^{1/3} = 3\sqrt[3]{2}$.

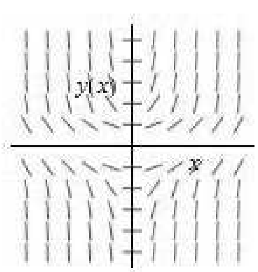
12. Find the integrating factor for the linear differential equation $x^4 \dfrac{dy}{dx} = 4x^3 y + x^2$.

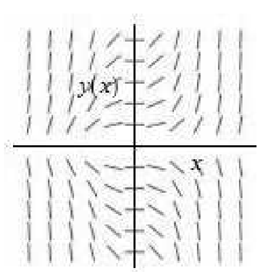
a. x^4
b. $4\ln x$
c. $-4\ln x$
d. $e^{-4/x}$
e. x^{-4} correct choice

Solution: In standard form the equation is $\dfrac{dy}{dx} - \dfrac{4}{x}y = \dfrac{1}{x^2}$.
So $P = -\dfrac{4}{x}$, $\int P\,dx = -\int \dfrac{4}{x}\,dx = -4\ln x$ and $I = e^{\int P\,dx} = e^{-4\ln x} = x^{-4}$

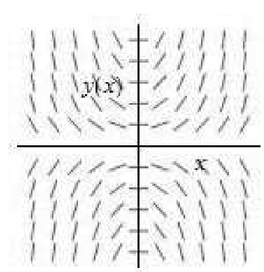
13. Which of the following is the direction field of the differential equation $\dfrac{dy}{dx} = x^2 y$?

a.


b.


c.


correct choice

d.


Solution: The slope, $x^2 y$, must be positive in the 1st and 2nd quadrants and negative in the 3rd and 4th quadrants.

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (20 points) For each series, determine if it is convergent or divergent.
Be sure to identify the Convergence Test(s) and check out their hypotheses.

a. (6 pts) $\sum_{n=2}^{\infty} n e^{\left(-n^2\right)}$

Solution: Use ***Integral Test***:

Consider $f(x) = x e^{\left(-x^2\right)}$

Note: The terms are positive, f interpolates: $f(n) = n e^{\left(-n^2\right)}$ and

f is decreasing because $f'(x) = e^{\left(-x^2\right)} - x e^{\left(-x^2\right)} 2x = e^{\left(-x^2\right)}(1 - 2x^2) < 0$ for $x \geq 2$.

We integrate using the substitution $u = -x^2 \qquad du = -2x\,dx$

$\int_2^{\infty} x e^{\left(-x^2\right)}\,dx = -\frac{1}{2}\int e^u\,du = -\frac{1}{2} e^{\left(-x^2\right)}\Big|_2^{\infty} = \frac{1}{2} e^{-4}$ which converges

So $\sum_{n=2}^{\infty} n e^{\left(-n^2\right)}$ converges.

b. (6 pts) $\sum_{n=2}^{\infty} \frac{n^5 - 3}{n^6 + 1}$

Solution: Compare to $\sum_{n=2}^{\infty} \frac{1}{n}$ which is a divergent harmonic series (p-series with $p = 1$).

Note: The Simple Comparison Test will not work because $\frac{n^5 - 3}{n^6 + 1} < \frac{n^5}{n^6} = \frac{1}{n}$

Use ***Limit Comparison Test***:

$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^5 - 3}{n^6 + 1} \cdot \frac{n}{1} = 1$ which is finite and non-zero.

So $\sum_{n=2}^{\infty} \frac{n^5 - 3}{n^6 + 1}$ diverges also.

c. (8 pts) $\sum_{n=2}^{\infty} \frac{\sin n}{n^2 + 1}$

Solution: The series is not positive and is not alternating. So we cannot use a test for positive series nor the alternating series test.

Use ***Related Absolute Series Test***:

The related absolute series is $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2 + 1}$ which is $< \sum_{n=2}^{\infty} \frac{1}{n^2}$ which is a p-series with $p = 2$ and so $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent. Consequently, $\sum_{n=2}^{\infty} \frac{|\sin n|}{n^2 + 1}$ is convergent by the comparison test. Consequently, $\sum_{n=2}^{\infty} \frac{\sin n}{n^2 + 1}$ converges because its related absolute series converges.

15. (16 points) Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n\,3^n}(x-4)^n$.

Be sure to identify the Convergence Test(s) and check out their hypotheses.

Solution: Use ***Ratio Test***: $|a_n| = \frac{|x-4|^n}{n\,3^n}$ $\quad |a_{n+1}| = \frac{|x-4|^{n+1}}{(n+1)\,3^{n+1}}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{(n+1)\,3^{n+1}} \frac{n\,3^n}{|x-4|^n} = \frac{|x-4|}{3} \lim_{n\to\infty}\left(\frac{n}{n+1}\right) = \frac{|x-4|}{3}$$

The series is absolutely convergent when $\rho = \frac{|x-4|}{3} < 1$ or $|x-4| < 3$.

So the radius of convergence is $R = 3$ and the series is absolutely convergent on $(1,7)$.

We check the endpoints:

$x = 1$: $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,3^n}(1-4)^n = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent ***harmonic series***.

$x = 7$: $\quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n\,3^n}(7-4)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is convergent by the ***Alternating Series Test***.

So the interval of convergence is $1 < x \le 7$ or $(1,7]$.

16. (18 points) A salt water fish tank contains 20 liters of water with 700 grams of salt. In order to reduce the salt concentration, you pour in salt water with a concentration of 15 grams of salt per liter at 2 liters per minute. You keep the tank well mixed and drain the mixture at 2 liters per minute. Let $S(t)$ be the amount of salt (in grams) in the tank at time t (in minutes).

a. (6 pts) Write the differential equation and initial condition for $S(t)$.

Solution: $\frac{dS}{dt}\frac{\text{g}}{\text{min}} = 15\frac{\text{g}}{\text{L}} \cdot 2\frac{\text{L}}{\text{min}} - \frac{S\text{g}}{20\text{L}} \cdot 2\frac{\text{L}}{\text{min}}$ or $\frac{dS}{dt} = 30 - \frac{1}{10}S$

$S(0) = 700$

b. (9 pts) Solve the initial value problem for $S(t)$.

Solution Method 1: The equation is separable. Separate and integrate.

$\int \frac{dS}{30 - \frac{1}{10}S} = \int dt \qquad -10\ln\left|30 - \frac{1}{10}S\right| = t + C$

Solve: $\ln\left|30 - \frac{1}{10}S\right| = -\frac{t}{10} - \frac{C}{10} \qquad \left|30 - \frac{1}{10}S\right| = e^{-C/10}e^{-t/10}$

$30 - \frac{1}{10}S = Ae^{-t/10}$ where $A = \pm e^{-C/10} \qquad S = 300 - 10Ae^{-t/10}$

Use the initial condition: At $t = 0$, we have $S = 700$. So

$700 = 300 - 10A \qquad A = -40$

Substitute back: $S = 300 + 400e^{-t/10}$

Solution Method 2: The equation is linear: Put it in standard form: $\frac{dS}{dt} + \frac{1}{10}S = 30$

Find the integrating factor: $I = \exp\left(\int \frac{1}{10}dt\right) = e^{t/10}$

Multiply the standard equation by I: $e^{t/10}\frac{dS}{dt} + \frac{1}{10}e^{t/10}S = 30e^{t/10}$

Recognize the left side as the derivative of a product: $\frac{d}{dt}(e^{t/10}S) = 30e^{t/10}$

Integrate and solve: $e^{t/10}S = \int 30e^{t/10}\,dt = 300e^{t/10} + C \qquad S = 300 + Ce^{-t/10}$

Use the initial condition: At $t = 0$, we have $S = 700$. So

$700 = 300 + C \qquad C = 400$

Substitute back: $S = 300 + 400e^{-t/10}$

c. (3 pts) After how many minutes will the amount of salt in the tank drop to 400 grams?

Solution: Set $S = 400$ and solve for t:

$400 = 300 + 400e^{-t/10} \qquad 400e^{-t/10} = 100 \qquad e^{-t/10} = \frac{1}{4} \qquad \frac{-t}{10} = \ln\left(\frac{1}{4}\right)$

$t = -10\ln\left(\frac{1}{4}\right) = 10\ln 4$