Name________________________________

1-8	/40	10	/20
9	/15	11	/25
		Total	/100

MATH 152H Exam 2 Spring 2017

Sections 203/204 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the arc length of the curve $(x,y) = \left(\frac{1}{2}t^6, t^4\right)$ from $(0,0)$ to $\left(\frac{1}{2}, 1\right)$.

a. $\frac{10}{9}$

b. $\frac{5}{9}$

c. $\frac{61}{54}$ correct choice

d. $\frac{1}{54}$

e. $\frac{1}{6}$

Solution: $\frac{dx}{dt} = 3t^5 \qquad \frac{dy}{dt} = 4t^3$

$$L = \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^1 \sqrt{9t^{10} + 16t^6}\,dt = \int_0^1 t^3\sqrt{9t^4 + 16}\,dt$$

$$u = 9t^4 + 16 \qquad du = 36t^3\,dt \qquad \frac{1}{36}\,du = t^3\,dt$$

$$L = \frac{1}{36}\int_0^1 \sqrt{u}\,du = \frac{1}{36}\frac{2u^{3/2}}{3} = \left[\frac{1}{54}(9t^4 + 16)^{3/2}\right]_0^1 = \frac{1}{54}(25^{3/2} - 16^{3/2}) = \frac{1}{54}(125 - 64) = \frac{61}{54}$$

2. The parabola $y = x^2$ for $0 \le x \le \sqrt{2}$ is revolved about the y-axis. Find the surface area swept out.

a. $\frac{13\pi}{3}$ correct choice

b. $\frac{13\pi}{6}$

c. $\frac{13\pi}{9}$

d. $\frac{26\pi}{3}$

e. $\frac{26\pi}{9}$

Solution: $\frac{dy}{dx} = 2x$ radius: $r = x$

$$A = \int_0^{\sqrt{2}} 2\pi r\,ds = \int_0^{\sqrt{2}} 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = 2\pi\int_0^{\sqrt{2}} x\sqrt{1 + 4x^2}\,dx$$

$$u = 1 + 4x^2 \qquad du = 8x\,dx \qquad \frac{1}{8}\,du = x\,dx$$

$$A = \frac{2\pi}{8}\int \sqrt{u}\,du = \frac{\pi}{4}\frac{2u^{3/2}}{3} = \left[\frac{\pi}{6}(1 + 4x^2)^{3/2}\right]_0^{\sqrt{2}} = \frac{\pi}{6}(9^{3/2} - 1) = \frac{13\pi}{3}$$

3. Find the general partial fraction expansion of $f(x) = \dfrac{x-1}{(x^3+x)(x^4-1)}$.

a. $\dfrac{A}{x} + \dfrac{Bx+C}{x^2-1} + \dfrac{Dx+E}{x^2+1} + \dfrac{Fx+G}{(x^2+1)^2}$

b. $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{Dx+E}{(x^2+1)^2}$

c. $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{Cx+D}{(x^2+1)^2}$

d. $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1} + \dfrac{Dx+E}{x^2+1} + \dfrac{Fx+G}{(x^2+1)^2}$

e. $\dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$ correct choice

Solution: $f(x) = \dfrac{x-1}{x(x^2+1)(x^2-1)(x^2+1)} = \dfrac{1}{x(x+1)(x^2+1)^2}$

$f(x) = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

4. In the partial fraction expansion $\dfrac{36x}{x^4-81} = \dfrac{Ax+B}{x^2+9} + \dfrac{C}{x+3} + \dfrac{D}{x-3}$, which coefficient is INCORRECT?

a. $A = -2$

b. $B = 6$ correct choice

c. $C = 1$

d. $D = 1$

e. All of the above are correct.

Solution: Clear the denominator:

$$36x = (Ax+B)(x^2-9) + C(x^2+9)(x-3) + D(x^2+9)(x+3)$$

Plug in $x = 3$: $108 = (3A+B)(0) + C(9+9)(0) + D(9+9)(3+3)$

$108 = D(108)$ $D = 1$

Plug in $x = -3$: $-108 = (-3A+B)(0) + C(9+9)(-3-3) + D(9+9)(0)$

$-108 = C(-108)$ $C = 1$

Plug in $x = 0$: $0 = B(-9) + 1(9)(-3) + 1(9)(3) = -9B$

$B = 0$ $\Leftarrow\Leftarrow\Leftarrow$ INCORRECT

Coeff of x^3 : $0 = A + C + D = A + 1 + 1$ $A = -2$

5. The base of a solid is the region between the parabola $y = x^2$ and the line $y = 4$ and the crosssections perpendicular to the y-axis are squares. Find its volume

a. 4
b. 8
c. 16
d. 32 correct choice
e. 64

Solution: The slice at height y has width $w = 2x = 2\sqrt{y}$. This is the side of the square. So the area is $A(y) = w^2 = 4y$. So the volume is
$$V = \int_0^4 A(y)\,dy = \int_0^4 4y\,dy = \left[2y^2\right]_0^4 = 32$$

6. The region bounded by the curves $y = x^4$, $y = 0$ and $x = 3$ is revolved about the y-axis. Find the volume swept out.

a. $3^7\pi$
b. $3^5\pi$ correct choice
c. $3^3\pi$
d. $\frac{3^5}{5}\pi$
e. $\frac{486}{5}\pi$

Solution: We use an x-integral. So the slices are vertical. When these are revolved about the y-axis, they sweep out cylinders with radius $r = x$ and height $h = y = x^4$.
$$V = \int_0^3 2\pi rh\,dx = 2\pi\int_0^3 xx^4\,dx = 2\pi\frac{x^6}{6}\Big|_0^3 = \pi\frac{3^6}{3} = 3^5\pi$$

7. The region bounded by the curves $y = x^4$, $y = 0$ and $x = 3$ is revolved about the x-axis. Find the volume swept out.

a. $3^7\pi$ correct choice
b. $3^5\pi$
c. $3^3\pi$
d. $\frac{3^5}{5}\pi$
e. $\frac{486}{5}\pi$

Solution: We use an x-integral. So the slices are vertical. When these are revolved about the x-axis, they sweep out disks with radius $r = y = x^4$.
$$V = \int_0^3 \pi r^2\,dx = \pi\int_0^3 x^8\,dx = \pi\frac{x^9}{9}\Big|_0^3 = 3^7\pi$$

8. Compute $\int_0^3 \frac{1}{(25-x^2)^{3/2}}\,dx$.

a. $\frac{3}{4}$

b. $\frac{3}{16}$

c. $\frac{3}{25}$

d. $\frac{3}{100}$ correct choice

e. $\frac{3}{400}$

Solution: Let $x = 5\sin\theta$. Then $dx = 5\cos\theta\, d\theta$.

$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \int \frac{1}{(25-25\sin^2\theta)^{3/2}} 5\cos\theta\, d\theta = \frac{1}{25}\int \frac{\cos\theta}{(1-\sin^2\theta)^{3/2}}\,d\theta = \frac{1}{25}\int \frac{\cos\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{1}{25}\int \sec^2\theta\, d\theta = \frac{1}{25}\tan\theta$$

Draw a triangle with hypotenuse 5 and opposite side x.

Then the adjacent side is $\sqrt{25-x^2}$ and $\tan\theta = \frac{x}{\sqrt{25-x^2}}$. So

$$\int \frac{1}{(25-x^2)^{3/2}}\,dx = \frac{1}{25}\frac{x}{\sqrt{25-x^2}}$$

$$\int_0^3 \frac{1}{(25-x^2)^{3/2}}\,dx = \frac{1}{25}\left[\frac{x}{\sqrt{25-x^2}}\right]_0^3 = \frac{1}{25}\left(\frac{3}{4}\right) = \frac{3}{100}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. $\left(15 \text{ points}\right)$ A water trough is 10 feet long and its end is an isosceles triangle with vertex down with height 4 feet and width 2 feet. The trough is filled with water to a depth of 3 feet. Find the work done to pump the water out of the trough to a height of 1 foot above the top of the trough. Assume the weight density of the water is $\rho = 64\ \frac{\text{lb}}{\text{ft}^3}$

Solution: Measure y up from the bottom of the trough. Then using similar triangles, the horizontal slice of the triangular end at height y has width w satisfying $\frac{w}{y} = \frac{2}{4}$. So $w = \frac{y}{2}$.

Then the area of the slice of the trough at height y is $A = 10w = 5y$. And the volume of the slice with thickness dy is $dV = A\,dy = 5y\,dy$. So the weight of the water in this slice is

$$dF = \rho dV = 64 \cdot 5y\,dy = 320y\,dy$$

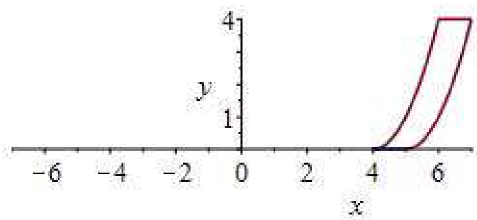
This slice of water must be lifted from height y to height 5 feet. So the distance it is lifted is $D = 5-y$. The water is between the heights 0 and 3. So the work done is

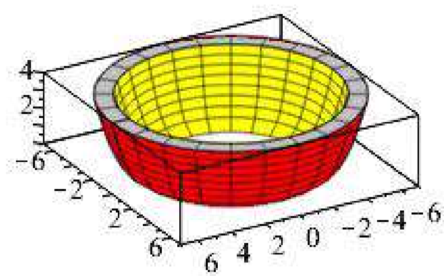
$$W = \int_0^3 D\,dF = \int_0^3 (5-y)\,320y\,dy = \int_0^3 (1600y - 320y^2)\,dy = \left[800y^2 - \frac{320}{3}y^3\right]_0^3$$

$$= 800 \cdot 9 - 320 \cdot 9 = 480 \cdot 9 = 4320 \text{ ft-lb.}$$

10. $\left(20 \text{ points}\right)$ The region between the curves $x = 4 + \sqrt{y}$ and $x = 5 + \sqrt{y}$ for $0 \le y \le 4$ (shown below), is rotated about the y-axis to form the clay bowl (also shown below). (Ignore the fact that there is no base.) In the rotated figure, y is the vertical axis, the inner radius is $r_1 = 4 + \sqrt{y}$ and the outer radius is $r_2 = 5 + \sqrt{y}$.

Here y is measured in *cm* and the density of the clay used to make the bowl is $\delta = \frac{3}{2} \frac{gm}{cm^3}$.





a. Find the volume of clay used to make the bowl.

Solution: This is a volume by slicing perpendicular to the y axis. Each horizontal crosssection is a washer of area

$$A(y) = \pi r_2{}^2 - \pi r_1{}^2 = \pi\left(5 + \sqrt{y}\right)^2 - \pi\left(4 + \sqrt{y}\right)^2$$
$$= \pi\left(25 + 10\sqrt{y} + y\right) - \pi\left(16 + 8\sqrt{y} + y\right) = \pi\left(9 + 2\sqrt{y}\right)$$

So the volume is

$$V = \int_0^4 A(y)\,dy = \pi \int_0^4 9 + 2\sqrt{y}\,dy = \pi\left[9y + \frac{4}{3}y^{3/2}\right]_0^4 = \pi\left(36 + \frac{4}{3}8\right) = \frac{140}{3}\pi$$

b. Find the mass of the clay used to make the bowl.

Solution: $M = \delta V = \frac{3}{2}\frac{140}{3}\pi = 70\pi$

c. Find the y-component of the center of mass of the bowl.

Solution:

$$M_1 = \int_0^4 y\delta A(y)\,dy = \pi\delta \int_0^4 y\left(9 + 2\sqrt{y}\right)dy = \pi\delta\left[9\frac{y^2}{2} + \frac{4}{5}y^{5/2}\right]_0^4$$
$$= \pi\frac{3}{2}\left(72 + \frac{4}{5}32\right) = \frac{3}{2}\frac{488}{5}\pi = \frac{732}{5}\pi$$

$$\bar{y} = \frac{732}{5}\pi\frac{1}{70\pi} = \frac{366}{175} = 2.09$$

11. (25 points) Given the partial fraction expansion $\dfrac{54x+54}{x^4-81} = \dfrac{2}{x-3} + \dfrac{1}{x+3} - \dfrac{3x+3}{x^2+9}$,

compute $\displaystyle\int \frac{54x+54}{x^4-81}\,dx.$

a. $\displaystyle\int \frac{2}{x-3}\,dx =$

Solution: $\displaystyle\int \frac{2}{x-3}\,dx = 2\ln|x-3| + C$

b. $\displaystyle\int \frac{1}{x+3}\,dx =$

Solution: $\displaystyle\int \frac{1}{x+3}\,dx = \ln|x+3| + C$

c. $\displaystyle\int \frac{-3x}{x^2+9}\,dx =$

Solution: Let $u = x^2+9 \qquad du = 2x\,dx \qquad x\,dx = \frac{1}{2}du$

$$\int \frac{-3x}{x^2+9}\,dx = \int \frac{-3}{2u}\,du = -\frac{3}{2}\ln|u| + C = -\frac{3}{2}\ln|x^2+9| + C$$

d. $\displaystyle\int \frac{-3}{x^2+9}\,dx =$

Solution: Let $x = 3\tan\theta \qquad dx = 3\sec^2\theta\,d\theta$

$$\int \frac{-3}{x^2+9}\,dx = \int \frac{-3}{9\tan^2\theta+9}\,3\sec^2\theta\,d\theta = \int -1\,d\theta = -\theta + C = -\arctan\frac{x}{3} + C$$

e. $\displaystyle\int \frac{54x+54}{x^4-81}\,dx =$

Solution: $\displaystyle\int \frac{54x+54}{x^4-81}\,dx = 2\ln|x-3| + \ln|x+3| - \frac{3}{2}\ln|x^2+9| - \arctan\frac{x}{3} + C$