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| MATH 171 | Exam 3 | Spring 2004 |
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| Sections 502 | Solutions | P. Yasskin |

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|---|---|
| 1-10 | /60 |
| 11 | /15 |
| 12 | /10 |
| 13 | /15 |
| Total | /100 |

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID and "Exam 3."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (6 points each. No part credit.)

**1.** Find the absolute minimum and absolute maximum values of the function $f(x) = x^3 - \frac{9}{2}x^2 + 14$ on the interval $[-2,4]$.

**a.** minimum $= -12$, maximum $= 6$

**b.** minimum $= -12$, maximum $= 14$ correctchoice

**c.** minimum $= -12$, maximum $= \frac{1}{2}$

**d.** minimum $= \frac{1}{2}$, maximum $= 14$

**e.** minimum $= \frac{1}{2}$, maximum $= 6$

$f(x) = x^3 - \frac{9}{2}x^2 + 14$ $\quad f'(x) = 3x^2 - 9x = 3x(x-3) = 0$
critical points at $x = 0,3$ endpoints at $x = -2,4$
$f(-2) = -12$ $\quad f(0) = 14$ $\quad f(3) = \frac{1}{2}$ $\quad f(4) = 6$ minimum $= -12$, maximum $= 14$

**2.** Where is $f(x) = \frac{1}{4}x^4 - 6x^2$ concave up?

**a.** $x < -2$ or $x > 2$ correctchoice

**b.** $-2 < x < 2$

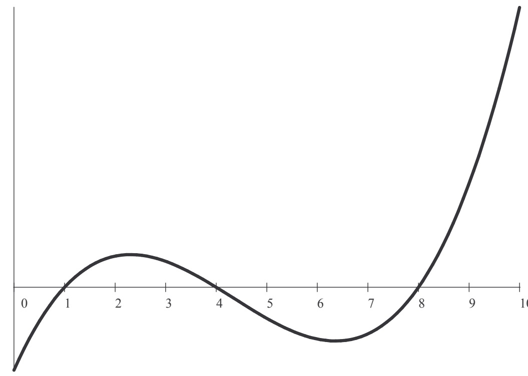
**c.** $x < -\sqrt{12}$ or $0 < x < \sqrt{12}$

**d.** $-\sqrt{12} < x < 0$ or $x > \sqrt{12}$

**e.** $-\sqrt{12} < x < \sqrt{12}$

$f'(x) = x^3 - 12x$ $\quad f''(x) = 3x^2 - 12 = 3(x-2)(x+2) > 0$ for $x < -2$ or $x > 2$.

**3.** At the right is the graph of $y = f'(x)$.
Where is $f(x)$ increasing?




**a.** $[0, 2.5]$ and $[6.5, 10]$

**b.** $[0, 1]$ and $[4, 8]$

**c.** $[2.5, 6.5]$

**d.** $[1, 4]$ and $[8, 10]$ correctchoice

**e.** $[4.5, 10]$

$f(x)$ is increasing where $f'(x) > 0$, i.e. on $[1, 4]$ and $[8, 10]$.

**4.** The graph of $y = f'(x)$ appears in the previous problem. Where does $f(x)$ have a local maximum?

**a.** $1$ and $8$ only

**b.** $2.5$ only

**c.** $4$ only correctchoice

**d.** $1, \quad 4$ and $8$ only

**e.** $2.5$ and $10$ only

$f(x)$ has a local maximum where $f'(x)$ switches from positive to negative, i.e. at $x = 4$.

**5.** Find all critical points of the function $f(x) = \sin x + \cos x$ on the interval $[0, 2\pi]$.

**a.** $\frac{\pi}{4}$ and $\frac{5\pi}{4}$ only correctchoice

**b.** $\frac{\pi}{4}$ only

**c.** $\frac{5\pi}{4}$ only

**d.** $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$ only

**e.** $\frac{3\pi}{4}$ only

$f'(x) = \cos x - \sin x = 0 \quad \Rightarrow \quad \sin x = \cos x \quad \Rightarrow \quad \tan x = 1 \quad \Rightarrow \quad x = \frac{\pi}{4} + n\pi$

**6.** A rocket starts at rest at ground level. It's vertical acceleration is $a(t) = 64\pi\sin(2t) - 32$ ft/sec$^2$ where $t$ is in sec. What is it's velocity at $t = \frac{\pi}{2}$ sec?

**a.** $16\pi$

**b.** $32\pi$

**c.** $48\pi$ correctchoice

**d.** $64\pi$

**e.** $72\pi$

$v(t) = -32\pi\cos(2t) - 32t + C \qquad v(0) = -32\pi + C = 0 \qquad C = 32\pi \qquad v(t) = -32\pi\cos(2t) - 32t + 32\pi$

$v\left(\frac{\pi}{2}\right) = -32\pi\cos\left(2\frac{\pi}{2}\right) - 32\frac{\pi}{2} + 32\pi = 32\pi - 16\pi + 32\pi = 48\pi$

**7.** Compute $\int_0^1 e^x\,dx$.

**a.** $e^2$

**b.** $\frac{e^2}{2}$

**c.** $\frac{e^2}{2} - e$

**d.** $e$

**e.** $e - 1$ correctchoice

$\int_0^1 e^x\,dx = [e^x]_0^1 = e^1 - e^0 = e - 1$

**8.** Find the area under $y = x^3$ between $x = 0$ and $x = 4$.

**a.** 8

**b.** 16

**c.** 32

**d.** 64 correctchoice

**e.** 128

$$A = \int_0^4 x^3\,dx = \left[\frac{x^4}{4}\right]_0^4 = 4^3 = 64$$

**9.** Compute $\int_0^1 x^2 \sin(4x^3)\,dx$.

**a.** $12 - 12\cos 4$

**b.** $12\cos 1 - 12$

**c.** $\frac{1}{12} - \frac{1}{12}\cos 4$ correctchoice

**d.** $\frac{1}{12}\cos 4 - \frac{1}{12}$

**e.** $\frac{1}{12}\cos 1 - \frac{1}{12}$

$$u = 4x^3 \qquad du = 12x^2\,dx \qquad \frac{1}{12}\,du = x^2\,dx$$

$$\int_0^1 x^2\sin(4x^3)\,dx = \frac{1}{12}\int_0^4 \sin u\,du = \left[\frac{-1}{12}\cos u\right]_0^4 = \frac{-1}{12}\cos 4 + \frac{1}{12}\cos 0 = \frac{1}{12} - \frac{1}{12}\cos 4$$

**10.** Compute $\int_1^e \frac{(\ln x)^2}{x}\,dx$.

**a.** $\frac{1}{3}$ correctchoice

**b.** 2

**c.** $\frac{e^3 - 1}{3}$

**d.** $2e - 2$

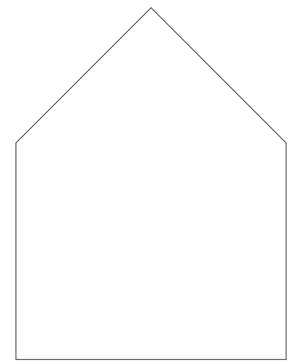
**e.** $\frac{2e - 2}{3}$

$$u = \ln x \qquad du = \frac{1}{x}\,dx \qquad \int_1^e \frac{(\ln x)^2}{x}\,dx = \int_0^1 u^2\,du = \frac{u^3}{3}\Big|_0^1 = \frac{1}{3}$$

Work Out: (Points indicated. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

**11.** (15 points) A church window will have the shape of a rectangle with an isosceles right triangle on top. The area of the window is to be $A = 1 + 2\sqrt{2}$. Since the window frame will be gold plated, we want to minimize the perimeter of the window. What are the dimensions of the rectangular part of the window which minimize the perimeter of the whole window?



You must explain your solution using sentences.

HINT: If $a$ is the height of the rectangle and $b$ is its width, then $\dfrac{b}{\sqrt{2}}$ is the length of each slanted side of the triangular top.

Let $a$ be the height and $b$ be the width of the rectangle.
The area is constrained to be $A = ab + \dfrac{b^2}{4} = 1 + 2\sqrt{2}$
and we wish to minimize the perimeter $P = 2a + b + 2b/\sqrt{2}$.
We solve the constraint for $a$: $a = \dfrac{1+2\sqrt{2}}{b} - \dfrac{b}{4}$ So the perimeter becomes:

$$P = 2\left(\frac{1+2\sqrt{2}}{b} - \frac{b}{4}\right) + b + \sqrt{2}\,b = \frac{2+4\sqrt{2}}{b} - \frac{b}{2} + b + \sqrt{2}\,b = \frac{2+4\sqrt{2}}{b} + \left(\frac{1}{2} + \sqrt{2}\right)b$$

We set the derivative equal to zero and solve for $b$:

$$P' = -\frac{2+4\sqrt{2}}{b^2} + \left(\frac{1}{2} + \sqrt{2}\right) = 0 \quad \Rightarrow \quad \left(\frac{1}{2} + \sqrt{2}\right) = \frac{2+4\sqrt{2}}{b^2}$$

$$\Rightarrow \quad b^2 = \frac{2+4\sqrt{2}}{\frac{1}{2} + \sqrt{2}} = \frac{4+8\sqrt{2}}{1+2\sqrt{2}} = 4$$

So $b = 2$ and $a = \dfrac{1+2\sqrt{2}}{2} - \dfrac{2}{4} = \sqrt{2}$

**12.** $(10$ points$)$ Let $P_n$ denote the statement: $\displaystyle\sum_{i=1}^{n} 3^i = \frac{3^{n+1}-3}{2}$

Use mathematical induction to prove $P_n$ is true for all integers $n \geq 1$.

**Initialization Step:** $P_1$ means $\displaystyle\sum_{i=1}^{1} 3^i = \frac{3^{1+1}-3}{2}$.

$\displaystyle\sum_{i=1}^{1} 3^i = 3^1 = 3$ and $\displaystyle\frac{3^{1+1}-3}{2} = \frac{9-3}{2} = 3$ So $P_1$ is true.

**Induction Step:** $P_k$ means $\displaystyle\sum_{i=1}^{k} 3^i = \frac{3^{k+1}-3}{2}$ and $P_{k+1}$ means $\displaystyle\sum_{i=1}^{k+1} 3^i = \frac{3^{k+2}-3}{2}$.

We assume $P_k$ is true. Then

$$\sum_{i=1}^{k+1} 3^i = \left(\sum_{i=1}^{k} 3^i\right) + 3^{k+1} = \left(\frac{3^{k+1}-3}{2}\right) + 3^{k+1} = \frac{3^{k+1}-3+2\cdot 3^{k+1}}{2} = \frac{3\cdot 3^{k+1}-3}{2} = \frac{3^{k+2}-3}{2}$$

So $P_{k+1}$ is true.

We conclude $P_n$ is true for all integers $n \geq 1$.

**13.** $(15$ points$)$ Use the Method of Riemann Sums with Right Endpoints

to compute the integral $\displaystyle\int_1^4 (x-1)^2\,dx$. Use the F.T.C. only to check your answer.

Hints: $\displaystyle\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

$$\Delta x = \frac{4-1}{n} = \frac{3}{n} \qquad x_i = 1 + i\Delta x = 1 + \frac{3i}{n} \qquad f(x) = (x-1)^2 \qquad f(x_i) = \left(1 + \frac{3i}{n} - 1\right)^2 = \frac{9i^2}{n^2}$$

$$\int_1^4 (x-1)^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{9i^2}{n^2}\frac{3}{n} = \lim_{n\to\infty} \frac{27}{n^3} \sum_{i=1}^{n} i^2 = \lim_{n\to\infty} \frac{27}{n^3}\frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n\to\infty} \frac{9}{2}\frac{(n+1)(2n+1)}{n^2} = \lim_{n\to\infty} \frac{9}{2}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right) = 9$$

Check: $\displaystyle\int_1^4 (x-1)^2\,dx = \left[\frac{(x-1)^3}{3}\right]_1^4 = \frac{3^3}{3} - \frac{0}{3} = 9.$