1. For what value(s) of $p$ are the vectors $\vec{a} = (3, p)$ and $\vec{b} = (4, 6)$, perpendicular?

a. 2 or $-2$ only
b. 2 only
c. $-2$ only correctchoice
d. $\frac{1}{2}$ only
e. $-\frac{1}{2}$ only

$\vec{a}$ and $\vec{b}$ are perpendicular iff $\vec{a} \cdot \vec{b} = 0$. In this case, $\vec{a} \cdot \vec{b} = 12 + 6p = 0$. So $p = -2$.

2. For what value of $b$ does $\lim_{x \to 2} f(x)$ exist if $f(x) = \begin{cases} x + 3 & \text{if } x < 2 \\ 4 & \text{if } x = 2 \\ x^2 + b & \text{if } x > 2 \end{cases}$

a. 1 correctchoice
b. 2
c. 3
d. 4
e. No values of $b$.

$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x + 3) = 5$ \quad $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^2 + b) = 4 + b$

So $\lim_{x \to 2} f(x)$ exist iff $4 + b = 5$, i.e. $b = 1$. 
3. Compute \( \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9} \)
   
   a. \( \frac{1}{2} \)
   
   b. \( \frac{1}{3} \) correct choice
   
   c. \( \frac{1}{6} \)
   
   d. \( \frac{2}{3} \)
   
   e. \( \frac{5}{6} \)
   
   \[ \lim_{x \to 3} \frac{x^2 - 4x + 3}{x^2 - 9} = \lim_{x \to 3} \frac{(x - 3)(x - 1)}{(x - 3)(x + 3)} = \lim_{x \to 3} \frac{x - 1}{x + 3} = \frac{2}{6} = \frac{1}{3} \]

4. Compute \( \lim_{x \to 0} \frac{\sin x - x}{x^3} \)
   
   a. \( -\frac{1}{6} \) correct choice
   
   b. \( -\frac{1}{3} \)
   
   c. 0
   
   d. \( \frac{1}{3} \)
   
   e. undefined
   
   \[ \lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0} \frac{\cos x - 1}{3x^2} = \lim_{x \to 0} \frac{-\sin x}{6x} = -\frac{1}{6} \]

5. As \( x \to \infty \), the function \( f(x) = \sqrt{x^2 + 5x} - \sqrt{x^2 + 2x} \) has a horizontal asymptote at
   
   a. \( -\frac{3}{2} \)
   
   b. \( -\frac{1}{2} \)
   
   c. 0
   
   d. \( \frac{1}{2} \)
   
   e. \( \frac{3}{2} \) correct choice
   
   \[ \lim_{x \to \infty} \frac{\sqrt{x^2 + 5x} - \sqrt{x^2 + 2x}}{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}} = \lim_{x \to \infty} \left( \frac{\sqrt{x^2 + 5x} - \sqrt{x^2 + 2x}}{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}} \right) \cdot \frac{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}}{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}} = \lim_{x \to \infty} \frac{(x^2 + 5x) - (x^2 + 2x)}{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}} = \lim_{x \to \infty} \frac{3x}{\sqrt{x^2 + 5x} + \sqrt{x^2 + 2x}} \cdot \frac{1}{x} = \lim_{x \to \infty} \frac{3}{\sqrt{1 + \frac{5}{x}} + \sqrt{1 + \frac{2}{x}}} = \frac{3}{2} \]
6. If \( f(x) = \ln(x^2 + x) \) then \( f'(2) = \)

a. \( \frac{1}{6} \)

b. \( \frac{1}{3} \)

c. \( \frac{1}{2} \)

d. \( \frac{2}{3} \)

e. \( \frac{5}{6} \) correctchoice

\[
f'(x) = \frac{2x + 1}{x^2 + x} \quad f'(2) = \frac{5}{6}
\]

7. If Pete is walking up a hill whose slope is \( 0.2 \) and his horizontal velocity is \( \frac{dx}{dt} = 6 \) mi/hr, what is his vertical velocity, \( \frac{dy}{dt} \)?

a. 30 mi/hr

b. 0.033 mi/hr

c. 0.833 mi/hr

d. 1.2 mi/hr correctchoice

e. 6.2 mi/hr

By chain rule, \( \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = 0.2 \cdot 6 = 1.2 \) mi/hr.

8. \( x = 2 \) is a critical point of the function \( f(x) = \frac{1}{4}x^4 - 2x^3 + 6x^2 - 8x \).

By the Second Derivative Test, \( x = 2 \) is

a. a local minimum.

b. a local maximum.

c. an inflection point.

d. The Second Derivative Test FAILS. correctchoice

\[
f'(x) = x^3 - 6x^2 + 12x - 8 \quad f''(x) = 3x^2 - 12x + 12 \quad f''(2) = 12 - 24 + 12 = 0 \quad \text{Test Fails.}
\]
9. If a rocket starts at \( x(0) = 0 \) m, with velocity \( v(0) = 1 \) m/sec, and accelerates at \( a(t) = 4e^{-2t} \) m/sec\(^2\), what is its position at \( t = 1 \) sec?

a. \( 2 - 16e^{-2} \)

b. \( 2 + e^{-2} \) correctchoice

c. 3

d. \( 1 + e^{-2} \)

e. \( 2 + 16e^{-2} \)

\[ v(t) = -2e^{-2t} + C \quad v(0) = -2 + C = 1 \quad C = 3 \quad v(t) = -2e^{-2t} + 3 \]

\[ x(t) = e^{-2t} + 3t + D \quad x(0) = 1 + D = 0 \quad D = -1 \quad x(t) = e^{-2t} + 3t - 1 \quad x(1) = e^{-2} + 3 - 1 = 2 + e^{-2} \]

10. Compute \( \int_{1/2}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx \)

a. \( \frac{\pi}{12} \)

b. \( \frac{\pi}{6} \)

c. \( \frac{\pi}{4} \)

d. \( \frac{\pi}{3} \) correctchoice

e. \( \frac{\pi}{2} \)

\[ \int_{1/2}^{1} \frac{1}{\sqrt{1 - x^2}} \, dx = \left[ \arcsin x \right]_{1/2}^{1} = \arcsin 1 - \arcsin \frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3} \]

11. Compute \( \int_{0}^{\pi} e^{\cos x} \sin x \, dx \)

a. 0

b. \( \frac{1}{e} - e \)

c. \( e - \frac{1}{e} \) correctchoice

d. \( -\frac{1}{e} \)

e. \( -e \)

\[ u = \cos x \quad du = -\sin x \, dx \]

\[ \int_{0}^{\pi} e^{\cos x} \sin x \, dx = -\int e^{u} \, du = -e^{u} = \left[ -e^{\cos x} \right]_{0}^{\pi} = -e^{-1} + e^{1} = e - \frac{1}{e} \]
Work Out: (10 points each. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

12. Find the equation of the line tangent to \( y = x^2 \) at the general point \( x = a \).
   For what value(s) of \( a \) does the tangent line pass through the point \( (3,8) \)?

   \[
   f(x) = x^2 \quad f(a) = a^2 \quad f'(x) = 2x \quad f'(a) = 2a \\
   y = f_{\text{tan}}(x) = f(a) + f'(a)(x-a) = a^2 + 2a(x-a) = 2ax - a^2 \\
   \]
   \( (3,8) \) lies on the tangent line if \( 8 = 2a^3 - a^2 \), or \( a^2 - 6a + 8 = 0 \), or \( (a-2)(a-4) = 0 \)
   So \( a = 2 \) or \( a = 4 \).

13. The area of a rectangle is held constant at 36 cm\(^2\) while the length and width are changing. If the length is currently 3 cm and is increasing at 2 cm/min, what is the width, is it increasing or decreasing and at what rate? Write your answer using sentences.

   Let \( l \) be the length and \( w \) be the width of the rectangle.

   Then the area is held constant at \( A = lw = 36 \). Solving for the width we find \( w = \frac{36}{l} \).

   Differentiating and using the chain rule, we find \( \frac{dW}{dt} = -\frac{36}{l^2} \frac{dl}{dt} \).

   Currently, \( l = 3 \) and \( \frac{dl}{dt} = 2 \). So currently the width is \( w = \frac{36}{3} = 12 \) cm,
   and it is changing at \( \frac{dW}{dt} = -\frac{36}{3^2}2 = -8 \) cm/min. So it is decreasing at 8 cm/min.

14. Determine exactly how many real solutions there are to the equation \( x^{12} + x^4 + x^2 - 2 = 0 \).
   Use sentences and name any theorems you use.

   Hint: Factor an \( x \) out of the derivative.

   Let \( f(x) = x^{12} + x^4 + x^2 - 2 \). Then \( f'(x) = 12x^{11} + 4x^3 + 2x = x(12x^{10} + 4x^2 + 2) \).

   The quantity in parentheses is always positive.

   So for \( x > 0 \) we have \( f'(x) > 0 \) and for \( x < 0 \) we have \( f'(x) < 0 \).

   By the Mean Value Theorem, \( f(x) \) is increasing for \( x > 0 \) and decreasing for \( x < 0 \).

   So there can be at most one solution for \( x > 0 \) and at most one solution for \( x < 0 \).

   We test some values:

   \[
   f(-1) = 1 + 1 + 1 - 2 = 1 \quad f(0) = -2 \quad f(1) = 1 + 1 + 1 - 2 = 1 \\
   \]
   Since \(-2 < 0 < 1\), by the Intermediate Value Theorem,
   there is at least one solution of \( f(x) = 0 \) on \([-1,0]\) and at least one solution on \([0,1]\).
   Therefore, there are exactly 2 solutions.
15. Find the dimensions and area of the largest rectangle that can be inscribed in the ellipse \(4x^2 + 9y^2 = 36.\)

Maximize \(A = 4xy\) subject to the constraint \(4x^2 + 9y^2 = 36.\)

Solve the constraint for \(y = \frac{1}{3}\sqrt{36 - 4x^2}\) and substitute into the area:

Maximize \(A = \frac{4}{3}x\sqrt{36 - 4x^2}\)

\[A' = \frac{4}{3}\sqrt{36 - 4x^2} + \frac{4}{3}x \cdot \frac{-4x}{\sqrt{36 - 4x^2}} = 0 \quad \Rightarrow \quad \sqrt{36 - 4x^2} = \frac{4x^2}{\sqrt{36 - 4x^2}}\]

\[\Rightarrow \quad 36 - 4x^2 = 4x^2 \quad \Rightarrow \quad 36 = 8x^2 \quad \Rightarrow \quad x^2 = \frac{36}{8} = \frac{9}{2} \quad \Rightarrow \quad x = \frac{3}{\sqrt{2}}\]

\[y = \frac{1}{3}\sqrt{36 - 4x^2} = \frac{1}{3}\sqrt{36 - 4 \cdot \frac{9}{2}} = \frac{1}{3}\sqrt{18} = \sqrt{2}\]

The dimensions are \(2x = \frac{6}{\sqrt{2}} = 3\sqrt{2}\) and \(2y = 2\sqrt{2}\) and the area is \(A = 4xy = 4 \cdot \frac{3}{\sqrt{2}} \cdot \sqrt{2} = 12.\)

16. Use the Method of Riemann Sums with Right Endpoints to compute the integral \(\int_{2}^{7} 8(x - 2)^3 \, dx.\)

Use the F.T.C. only to check your answer.

Hints: \(\sum_{i=1}^{n} 1 = n\) \(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\) \(\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}\) \(\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2\)

\(\Delta x = \frac{7 - 2}{n} = \frac{5}{n}\) \(x_i = 2 + i\Delta x = 2 + \frac{5i}{n}\) \(f(x) = 8(x - 2)^3\) \(f(x_i) = 8\left(2 + \frac{5i}{n} - 2\right)^3 = \frac{1000i^3}{n^3}\)

\[\int_{2}^{7} 8(x - 2)^3 \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} \frac{1000i^3}{n^3} \cdot \frac{5}{n} = \lim_{n \to \infty} \frac{5000}{n^4} \sum_{i=1}^{n} i^3 = \lim_{n \to \infty} \frac{5000}{n^4} \left(\frac{n(n+1)}{2}\right)^2\]

\[= \lim_{n \to \infty} 5000\left(\frac{n(n+1)}{2n^2}\right)^2 = \lim_{n \to \infty} 5000\left(\frac{(n+1)}{2n}\right)^2 = \lim_{n \to \infty} 5000\left(\frac{1}{2} + \frac{1}{2n}\right)^2 = \frac{5000}{4} = 1250\]

Check: \(\int_{2}^{7} 8(x - 2)^3 \, dx = \left[2(x - 2)^4\right]_{2}^{7} = 2(5)^4 = 1250.\)