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MATH 172 EXAM 2 Fall 1998

Section 502 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Compute $\int_1^4 \frac{\ln x}{\sqrt{x}}\, dx$

a. $4\ln 4$
b. $4\ln 4 - 2$
c. $4\ln 4 - 4$ correctchoice
d. $4\ln 4 - 12$
e. $2\ln 2 - 8$

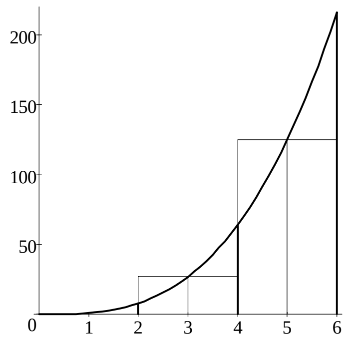
Integration by parts:

$$u = \ln x \qquad dv = \frac{1}{\sqrt{x}}\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = 2\sqrt{x}$$

$$\int_1^4 \frac{\ln x}{\sqrt{x}}\, dx = [2\sqrt{x}\ln x]_1^4 - \int_1^4 \frac{2\sqrt{x}}{x}\, dx = [2\sqrt{x}\ln x]_1^4 - \int_1^4 \frac{2}{\sqrt{x}}\, dx$$
$$= [2\sqrt{x}\ln x]_1^4 - [4\sqrt{x}\,]_1^4 = 2\sqrt{4}\ln 4 - 2\sqrt{1}\ln 1 - 4\sqrt{4} + 4\sqrt{1} = 4\ln 4 - 4$$

2. Approximate the integral $\int_0^6 x^3\, dx$ by using the Midpoint Rule with 3 rectangles with equal widths

a. 162
b. 306 correctchoice
c. 324
d. 360
e. 648



With $f(x) = x^3$ we have

$$\int_0^6 x^3\, dx \approx 2f(1) + 2f(3) + 2f(5)$$
$$\approx 2(1 + 27 + 125) = 306$$

3. Compute $\int_0^{\pi/2} \sin^4\theta \cos\theta \; d\theta$

a. $-\frac{1}{3}$

b. $-\frac{1}{5}$

c. $\frac{1}{6}$

d. $\frac{1}{5}$ correctchoice

e. $\frac{1}{3}$

$$\underbrace{u = \sin\theta \qquad du = \cos\theta \; d\theta}_{\text{May be skipped.}} \qquad \int_0^{\pi/2} \sin^4\theta \cos\theta \; d\theta = \underbrace{\int_{\theta=0}^{\pi/2} u^4 \; du = \left[\frac{u^5}{5} \right]_{\theta=0}^{\pi/2}}_{\text{May be skipped.}} = \left[\frac{\sin^5\theta}{5} \right]_0^{\pi/2} = \frac{1}{5}$$

4. Compute $\int_0^{\pi/4} \tan^3\theta \sec^2\theta \; d\theta$

a. $-\frac{1}{4}$

b. $\frac{1}{4}$ correctchoice

c. $-\frac{1}{3}$

d. $\frac{1}{3}$

e. $-\frac{1}{2}$

$$\underbrace{u = \tan\theta \qquad du = \sec^2\theta \; d\theta}_{\text{May be skipped.}} \qquad \int_0^{\pi/4} \tan^3\theta \sec^2\theta \; d\theta = \underbrace{\int_{\theta=0}^{\pi/4} u^3 \; du = \left[\frac{u^4}{4} \right]_{\theta=0}^{\pi/4}}_{\text{May be skipped.}} = \left[\frac{\tan^4\theta}{4} \right]_0^{\pi/4} =$$

5. Compute $\int_e^{\infty} \frac{1}{x(\ln x)^2} dx$

a. $-\infty$

b. $-\frac{1}{e}$

c. $\frac{1}{e}$

d. 1 correctchoice

e. ∞

$$u = \ln x \qquad du = \frac{1}{x} \; dx$$

$$\int_e^{\infty} \frac{1}{x(\ln x)^2} dx = \int_{x=e}^{\infty} \frac{1}{u^2} \; du = \lim_{b\to\infty}\left[-\frac{1}{u} \right]_{x=e}^{b} = \lim_{b\to\infty}\left[-\frac{1}{\ln x} \right]_e^b = \lim_{b\to\infty}\left[-\frac{1}{\ln b} + \frac{1}{\ln e} \right] = 1$$

6. Compute $\int_1^{e} \frac{1}{x(\ln x)^2} dx$

a. $-\infty$

b. -1

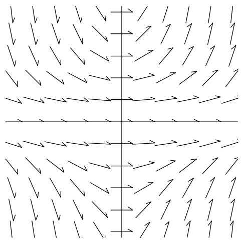
c. $\frac{1}{e}$

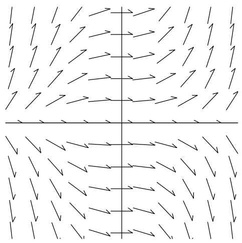
d. 1

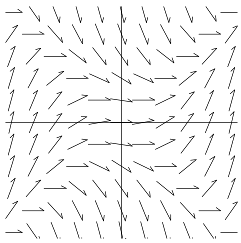
e. ∞ correctchoice

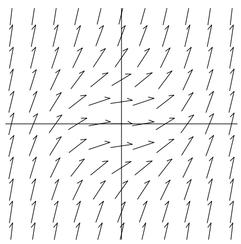
$$\int_1^{e} \frac{1}{x(\ln x)^2} dx = \int_{x=1}^{e} \frac{1}{u^2} \; du = \lim_{a\to 1^+}\left[-\frac{1}{u} \right]_{x=a}^{e} = \lim_{a\to 1^+}\left[-\frac{1}{\ln x} \right]_a^e = \lim_{a\to 1^+}\left[-\frac{1}{\ln e} + \frac{1}{\ln a} \right] = \infty$$

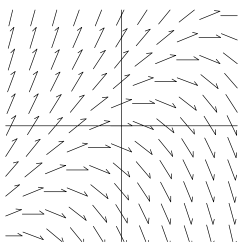
7. Which of the following is the direction field of the differential equation $\frac{dy}{dx} = x^2y$?

a.  No, $y' > 0$ in Quadrant II

b.  correctchoice

c.  No, $y' > 0$ in Quadrant I

d.  No, $y' < 0$ in Quadrant III

e.  No, $y' > 0$ in Quadrant I

8. Given that the partial fraction expansion of $\dfrac{1-x^2}{x^2(1+x^2)}$ is

$\dfrac{1-x^2}{x^2(1+x^2)} = \dfrac{1}{x^2} - \dfrac{2}{1+x^2}$, compute: $\displaystyle\int \frac{1-x^2}{x^2(1+x^2)}\,dx$

a. $-\dfrac{1}{2x} + C$

b. $-\dfrac{2}{x^3} + \dfrac{4x}{(1+x^2)^2} + C$

c. $\ln x^2 - \ln(1+x^2) + C$

d. $\ln x^2 - 2\ln(1+x^2) + C$

e. $-\dfrac{1}{x} - 2\arctan x + C$ correctchoice

$$\int \frac{1-x^2}{x^2(1+x^2)}\,dx = \int \frac{1}{x^2} - \frac{2}{1+x^2}\,dx = -\frac{1}{x} - 2\arctan x + C$$

9. Solve the differential equation $\dfrac{dy}{dx} = xy^2 + y^2$ with the initial condition $y(0) = 2$. Then find $y(1)$.

a. -1 correctchoice

b. 0

c. 1

d. $-\dfrac{2}{3}$

e. $\dfrac{3}{2}$

Separate variables: $\dfrac{dy}{dx} = y^2(x+1)$ $\quad \dfrac{dy}{y^2} = (x+1)\,dx$ $\quad \displaystyle\int \frac{dy}{y^2} = \int (x+1)\,dx$

$-\dfrac{1}{y} = \dfrac{x^2}{2} + x + C$ Initial Cond: $x = 0, y = 2$ $\quad -\dfrac{1}{2} = C$ $\quad -\dfrac{1}{y} = \dfrac{x^2}{2} + x - \dfrac{1}{2}$

Solve for y: $\dfrac{1}{y} = -\dfrac{x^2}{2} - x + \dfrac{1}{2}$ $\quad y = \dfrac{1}{-\frac{x^2}{2} - x + \frac{1}{2}}$ So: $y(1) = \dfrac{1}{-\frac{1}{2} - 1 + \frac{1}{2}} = -1$

10. Solve the differential equation $\dfrac{dy}{dx} - \dfrac{y}{x} = 2x^2$ with the initial condition $y(1) = 3$. Then find $y(2)$.

a. 2

b. 4

c. 6

d. 8

e. 12 correctchoice

Linear equation: $P(x) = -\dfrac{1}{x}$ $\quad Q(x) = 2x^2$

Integrating factor: $I = e^{\int P dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \dfrac{1}{x}$

Multiply equation by $I = \dfrac{1}{x}$: $\quad \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x^2}y = 2x$ $\quad \dfrac{d}{dx}\left(\dfrac{1}{x}y\right) = 2x$

Solve: $\dfrac{1}{x}y = \displaystyle\int 2x\,dx = x^2 + C$ Initial Cond: $x = 1, y = 3$ $\quad 3 = 1 + C$ $\quad C = 2$

$\dfrac{1}{x}y = x^2 + 2$ $\quad y = x^3 + 2x$ So: $y(2) = 2^3 + 2 \bullet 2 = 12$

11. (15 points) Compute $\int_0^1 (t^2 - t)\, e^{2t}\, dt$

$$\int_0^1 (t^2 - t)\, e^{2t}\, dt$$

Step 1: $u = t^2 - t \quad dv = e^{2t}\, dt$

$du = 2t - 1 \quad v = \frac{1}{2}e^{2t}$

$$= \left[(t^2 - t)\frac{1}{2}e^{2t} - \int \frac{2t-1}{2}e^{2t}\, dt \right]_0^1$$

Step 2: $u = \frac{2t-1}{2} \quad dv = e^{2t}\, dt$

$du = dt \quad v = \frac{1}{2}e^{2t}$

$$= \left[(t^2 - t)\frac{1}{2}e^{2t} - \frac{2t-1}{4}e^{2t} + \int \frac{1}{2}e^{2t}\, dt \right]_0^1$$

$$= \left[(t^2 - t)\frac{1}{2}e^{2t} - \frac{2t-1}{4}e^{2t} + \frac{1}{4}e^{2t} \right]_0^1$$

$$= \left[-\frac{2-1}{4}e^2 + \frac{1}{4}e^2 \right] - \left[-\frac{-1}{4} + \frac{1}{4} \right] = -\frac{1}{2}$$

12. (10 points) Compute $\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx$

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\, dx \qquad x = \sec\theta$$

$$= \int_0^{\pi/3} \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta}\, \sec\theta\tan\theta\, d\theta \qquad dx = \sec\theta\tan\theta\, d\theta$$

$$= \int_0^{\pi/3} \tan^2\theta\, d\theta$$

Change limits: $x = 1$ when $\theta = 0$

$$= \int_0^{\pi/3} \sec^2\theta - 1\, d\theta$$

$x = 2$ when $\sec\theta = 2$

$$= \left[\tan\theta - \theta \right]_0^{\pi/3}$$

or $\cos\theta = \frac{1}{2}$ or $\theta = \frac{\pi}{3}$

$$= \tan\frac{\pi}{3} - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}$$

OR substitute back:

$$= \left[\tan\theta - \theta \right]_{x=1}^{2} = \left[\sqrt{\sec^2\theta - 1} - \theta \right]_{x=1}^{2} = \left[\sqrt{x^2-1} - \operatorname{arcsec} x \right]_{x=1}^{2}$$

$$= \left[\sqrt{4-1} - \operatorname{arcsec} 2 \right] - \left[\sqrt{1-1} - \operatorname{arcsec} 1 \right] = \sqrt{3} - \frac{\pi}{3}$$

13. (10 points) Find the partial fraction expansion for $\dfrac{4x-6}{(x-1)^2(x^2+1)}$.

(Do not integrate.)

Hints: Try: $x = 1$, $x = 0$ and $\dfrac{d}{dx}$.

$$\frac{4x-6}{(x-1)^2(x^2+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{Cx+D}{(x^2+1)}$$

$4x - 6 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$

$x = 1:$ $\quad -2 = B(2)$ $\qquad B = -1$

$x = 0:$ $\quad -6 = A(-1) + B + D = -A - 1 + D$ $\qquad D = A - 5$

$\dfrac{d}{dx}:$ $\quad 4 = A[(x^2+1) + (x-1)(2x)] + B(2x) + C(x-1)^2 + (Cx+D)2(x-1)$

$x = 1:$ $\quad 4 = A(2) + B(2) = 2A - 2$ $\qquad 2A = 6$ $\qquad A = 3$

$D = A - 5 = 3 - 5$ $\qquad D = -2$

$x = 0:$ $\quad 4 = A(1) + C + D2(-1) = A + C - 2D$

$C = 4 - A + 2D = 4 - 3 + 2(-2)$ $\qquad C = -3$

So: $$\frac{4x-6}{(x-1)^2(x^2+1)} = \frac{3}{(x-1)} + \frac{-1}{(x-1)^2} + \frac{-3x-2}{(x^2+1)}$$

14. (15 points) A tank initially contains 12L of sugar water with 8gm of dissolved sugar. Sugar water that contains 2gm of sugar per liter is poured into the tank at the rate of 3L/min. The solution is kept thoroughly mixed and drains from the tank at the rate of 3L/min. How much sugar is in the tank

(a) after t min (b) after 16min and (c) asymptotically after a large time?

Let $s(t)$ be the grams of sugar in the tank at time t.

$$\frac{ds}{dt} = \underbrace{2\frac{\text{gm}}{\text{L}}3\frac{\text{L}}{\text{min}}}_{\text{IN}} - \underbrace{\frac{s(t)}{12}\frac{\text{gm}}{\text{L}}3\frac{\text{L}}{\text{min}}}_{\text{OUT}} \qquad \frac{ds}{dt} = 6 - \frac{1}{4}s$$

$$\int \frac{ds}{6-\frac{1}{4}s} = \int dt \qquad -4\ln\left(6-\frac{1}{4}s\right) = t + C$$

InitialCond: $s(0) = 8$ $\quad t = 0, s = 8$ $\quad -4\ln\left(6-\frac{1}{4}8\right) = C$ $\quad C = -4\ln 4$

Plug back and solve: $-4\ln\left(6-\frac{1}{4}s\right) = t - 4\ln 4$ $\quad \ln\left(6-\frac{1}{4}s\right) = -\frac{t}{4} + \ln 4$

$6 - \frac{1}{4}s = e^{-t/4+\ln 4} = 4e^{-t/4}$ $\qquad \frac{1}{4}s = 6 - 4e^{-t/4}$

(a) $s(t) = 24 - 16e^{-t/4}$

(b) $s(16) = 24 - 16e^{-4}$

(c) $\lim\limits_{t\to\infty} s(t) = 24$