MATH 172 EXAM 3 Fall 1998

Section 502 Solutions P. Yasskin

Multiple Choice: (5 points each)

1. Compute $\lim_{n\to\infty} \frac{3n^2}{1+n^3}$

- **a.** 0 correctchoice
- **b.** 1
- **c.** 2
- **d.** 3
- **e.** Divergent

$$\lim_{n\to\infty} \frac{3n^2}{1+n^3} = \lim_{n\to\infty} \frac{\frac{3}{n}}{\frac{1}{n^3}+1} = \frac{0}{1} = 0$$

2. Find r such that $5 + 5r + 5r^2 + 5r^3 + 5r^4 + \cdots = 3.$

- **a.** $\frac{2}{5}$
- **b.** $-\frac{2}{5}$
- **c.** $\frac{3}{5}$
- **d.** $\frac{5}{3}$
- **e.** $-\frac{2}{3}$ correctchoice

$$\frac{5}{1-r} = 3 \qquad \frac{5}{3} = 1-r \qquad r = 1 - \frac{5}{3} = -\frac{2}{3}$$

3. The series $\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$ is

- **a.** divergent by comparison to $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$.
- **b.** convergent by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$. correctchoice
- **c.** divergent by the ratio test.
- **d.** convergent by the ratio test.
- **e.** divergent by the n^{th}-term test.

In the series $\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$ the largest term in the denominator is n^2. So, we apply the Comparison Test by comparing with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series since $p = 2 > 1$. Since $n^2+\sqrt{n} > n^2$ we have $\frac{1}{n^2+\sqrt{n}} < \frac{1}{n^2}$. So $\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$ is also convergent.

4. Compute $\sum_{k=1}^{99}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)$

- **a.** .9 correctchoice
- **b.** .99
- **c.** 1
- **d.** 1.1
- **e.** Divergent

$$\sum_{k=1}^{99}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right) = \left(\frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \cdots + \left(\frac{1}{\sqrt{99}} - \frac{1}{\sqrt{100}}\right)$$

$$= 1 - \frac{1}{10} = .9$$

5. Compute $\sum_{n=1}^{\infty}\frac{3n^2}{1+n^3}$

- **a.** $\ln 2$
- **b.** $\frac{3}{2}$
- **c.** $\frac{27}{82}$
- **d.** Convergent but none of the above
- **e.** Divergent correctchoice

$\int_1^\infty \frac{3n^2}{1+n^3}\,dn = \ln(1+n^3)\Big|_1^\infty = \infty - \ln 2 = \infty$ So $\sum_{n=1}^{\infty}\frac{3n^2}{1+n^3}$ is divergent by the Integral Test.

6. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{3\sqrt{n}}$ is

- **a.** absolutely convergent.
- **b.** conditionally convergent. correctchoice
- **c.** absolutely divergent.
- **d.** conditionally divergent.
- **e.** oscillatory divergent.

The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{3\sqrt{n}}$ is convergent because it is an alternating, decreasing series and $\lim_{n\to\infty}\frac{1}{3\sqrt{n}} = 0.$ The related absolute series is $\sum_{n=1}^{\infty}\frac{1}{3\sqrt{n}} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$ which is a divergent p-series since $p = \frac{1}{2} < 1.$ So $\sum_{n=1}^{\infty}\frac{(-1)^n}{3\sqrt{n}}$ is conditionally convergent.

7. Compute $\lim_{x\to 0}\dfrac{\cos(2x)-1+2x^2}{x^4}$

- **a.** 0
- **b.** $\frac{1}{24}$
- **c.** $\frac{1}{12}$
- **d.** $\frac{2}{3}$ correctchoice
- **e.** ∞

$$\lim_{x\to 0}\frac{\cos(2x)-1+2x^2}{x^4} = \lim_{x\to 0}\frac{\left[1-\frac{(2x)^2}{2!}+\frac{(2x)^4}{4!}-\cdots\right]-1+2x^2}{x^4}$$

$$= \lim_{x\to 0}\frac{\frac{(2x)^4}{4!}-\cdots}{x^4} = \frac{16}{24} = \frac{2}{3}$$

8. Given that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ (for $|x|<1$), then (for $|x|<1$) we have $\sum_{n=0}^{\infty} nx^n =$

- **a.** $\frac{1}{1-x}$
- **b.** $\frac{1}{(1-x)^2}$
- **c.** $\frac{x}{(1-x)^2}$ correctchoice
- **d.** $\frac{x}{1-x}$
- **e.** $\frac{n}{1-x}$

Start with $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. Apply $\frac{d}{dx}$ to get $\sum_{n=0}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$.

Multiply by x to get $\sum_{n=0}^{\infty} nx^n = \frac{x}{(1-x)^2}$.

9. The series $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n$ converges to

- **a.** $\ln 2$
- **b.** $\sqrt{e}$ correctchoice
- **c.** $\sin\left(\frac{1}{2}\right)$
- **d.** $\sin(2)$
- **e.** e^2

$\sum_{n=0}^{\infty}\frac{1}{n!}(x)^n = e^x$ So: $\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n = e^{1/2} = \sqrt{e}$

10. Find the 3^{rd} degree term in the Taylor series for $f(x) = \frac{1}{x}$ centered at $x = 2$.

- **a.** $\frac{3}{8}(x-2)^3$
- **b.** $\frac{-3}{8}(x-2)^3$
- **c.** $-6(x-2)^3$
- **d.** $\frac{-1}{16}(x-2)^3$ correctchoice
- **e.** $\frac{1}{16}(x-2)^3$

$f'(x) = \frac{-1}{x^2} \quad f''(x) = \frac{2}{x^3} \quad f'''(x) = \frac{-6}{x^4} \quad f'''(2) = \frac{-6}{16} = -\frac{3}{8}$

So the 3^{rd} degree term is $\frac{f'''(2)}{3!}(x-2)^3 = \frac{1}{6}\left(-\frac{3}{8}\right)(x-2)^3 = -\frac{1}{16}(x-2)^3$.

11. (15 points) Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{(x-5)^n}{3^n n^3}$.

Be sure to identify each of the following and give reasons:

(1 pt) Center of Convergence: $a = \underline{\quad 5 \quad}$

$a_n = \frac{(x-5)^n}{3^n n^3} \qquad a_{n+1} = \frac{(x-5)^{n+1}}{3^{n+1}(n+1)^3}$

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)^{n+1}}{3^{n+1}(n+1)^3}\,\frac{3^n n^3}{(x-5)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-5)}{3}\left(\frac{n}{n+1}\right)^3\right| = \frac{|x-5|}{3}$$

The series converges if $\frac{|x-5|}{3} < 1$ or $|x-5| < 3$

Radius of Convergence: $R = \underline{\quad 3 \quad}$ (5 pt)

(1 pt) Right Endpoint: $x = \underline{\quad 5+3 = 8 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(8-5)^n}{3^n n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it is s p-series with $p = 3 > 1$.

At the Right Endpoint the Series [✓ Converges] Diverges (circle one) (3 pt)

(1 pt) Left Endpoint: $x = \underline{\quad 5-3 = 2 \quad}$

The series $\sum_{n=1}^{\infty} \frac{(2-5)^n}{3^n n^3} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}$ converges because its related absolute series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges OR because it is an alternating decreasing series and $\lim_{n\to\infty} \frac{1}{n^3} = 0$.

At the Left Endpoint the Series [✓ Converges] Diverges (circle one) (3 pt)

(1 pt) Interval of Convergence: $\underline{\quad 2 \le x \le 8 \quad \text{or} \quad [2,8] \quad}$

12. (15 points) Let $f(x) = x^2 \cos x$.

a. (10 pt) Find the Maclaurin series for $f(x)$. Write the series in summation form and also write out the first 4 terms.

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$x^2 \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n)!} = x^2 - \frac{x^4}{2} + \frac{x^6}{4!} - \frac{x^8}{6!} + \cdots$$

b. (5 pt) Find $f^{(6)}(0)$.

The term with an x^6 is $\frac{f^{(6)}(0)}{6!}x^6 = \frac{x^6}{4!}$. So: $f^{(6)}(0) = \frac{6!}{4!} = 30.$

13. (10 points) Given that $\ln(1+t) = t - \frac{1}{2}t^2 + \frac{1}{3}t^3 - \cdots,$ find the 6th degree Taylor polynomial approximation about $x = 0$ for $\ln(1+x^2)$.

Just substitute $t = x^2$: $\ln(1+x^2) = x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \cdots$

14. (15 points) You are given: $e^{(-x^2)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots.$

a. (10 pt) Use the quadratic Taylor polynomial approximation about $x = 0$ for $e^{(-x^2)}$ to estimate $\int_0^{0.1} e^{(-x^2)}\, dx$. (Keep 8 digits.)

The quadratic Taylor polynomial approximation is $e^{(-x^2)} = 1 - x^2$. So integrate:

$$\int_0^{0.1} e^{(-x^2)}\, dx = \int_0^{0.1} 1 - x^2 \, dx = \left[x - \frac{x^3}{3} \right]_0^{0.1} = .1 - \frac{.001}{3} = .09966667$$

b. (5 pt Extra Credit) Your result in (a) is equal to $\int_0^{0.1} e^{(-x^2)}\, dx$ to within ± how much? Why?

Since $e^{(-x^2)}$ and $\int_0^{0.1} e^{(-x^2)}\, dx$ are alternating decreasing series, the error is at most the next term:

$$\int_0^{0.1} \frac{x^4}{2}\, dx = \left[\frac{x^5}{10} \right]_0^{0.1} = \frac{(.1)^5}{10} = 10^{-6} = .000001$$