

Name _____ Sec _____

MATH 221 Exam 1 Fall 2009
 Section 503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1-11	/66	13	/10
12	/20	14	/10
		Total	/106

1. If $f(x,y) = x^2 \cos(y^2)$, which of the following is FALSE?

- a. $f_x(x,y) = 2x \cos(y^2)$
- b. $f_y(x,y) = -2x^2 y \sin(y^2)$
- c. $f_{xx}(x,y) = 2 \cos(y^2)$
- d. $f_{yy}(x,y) = -4x^2 y \cos(y^2)$ **Correct Choice**
- e. $f_{xy}(x,y) = -4xy \sin(y^2)$

$$f_{yy}(x,y) = \frac{d}{dy}[-2x^2 y \sin(y^2)] = -2x^2 \sin(y^2) - 4x^2 y^2 \cos(y^2)$$

2. The quadratic surface $x^2 - y^2 + z^2 - 4x - 6y - 10z + 16 = 0$ is a

- a. hyperboloid of 1 sheet and center (2,3,5)
- b. hyperboloid of 1 sheet and center (2,-3,5) **Correct Choice**
- c. hyperboloid of 2 sheets and center (2,3,5)
- d. hyperboloid of 2 sheets and center (2,-3,5)
- e. cone with vertex (2,3,5)

$$(x^2 - 4x) - (y^2 + 6y) + (z^2 - 10z) + 16 = 0$$

$$(x^2 - 4x + 4) - (y^2 + 6y + 9) + (z^2 - 10z + 25) + 16 = 4 - 9 + 25$$

$$(x - 2)^2 - (y + 3)^2 + (z - 5)^2 = 4 \quad \boxed{\text{hyperboloid with center } (2, -3, 5)}$$

$$(x - 2)^2 + (z - 5)^2 = 4 + (y + 3)^2 \quad (x - 2)^2 + (z - 5)^2 \geq 4 \quad \boxed{1 \text{ sheet}}$$

3. An airplane is travelling due North at constant speed and a constant altitude as it crosses the equator. In what direction does the \hat{B} vector point?

HINTS: Remember the Earth is curved. Ignore the rotation of the Earth.

- a. East
- b. West **Correct Choice**
- c. South
- d. Up
- e. Down

\hat{T} points North. \hat{N} points Down. So $\hat{B} = \hat{T} \times \hat{N}$ points West.

4. A triangle has edge vectors $\vec{AB} = (2, 1, -2)$ and $\vec{AC} = (-2, -2, 4)$. Find the altitude of the triangle if \vec{AB} is the base.

- a. $\frac{2\sqrt{5}}{3}$ Correct Choice
 b. $\frac{\sqrt{5}}{3}$
 c. $2\sqrt{5}$
 d. $\sqrt{5}$
 e. $3\sqrt{5}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ -2 & -2 & 4 \end{vmatrix} = \hat{i}(4 - 4) - \hat{j}(8 - 4) + \hat{k}(-4 + 2) = (0, -4, -2)$$

$$\text{Area} = \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2} \sqrt{(-4)^2 + (-2)^2} = \frac{1}{2} \sqrt{20} = \sqrt{5}$$

$$\text{Area} = \frac{1}{2} \text{Base} \cdot \text{Altitude} \quad \text{Base} = |\vec{AB}| = \sqrt{4 + 1 + 4} = 3 \quad \text{Altitude} = \frac{2\text{Area}}{\text{Base}} = \frac{2\sqrt{5}}{3}$$

5. A box slides down the helical ramp $\vec{r}(t) = (4 \cos t, 4 \sin t, 9 - 3t)$ starting at height $z = 9$ and ending at height $z = 0$. How far does the box slide?
- a. 3
 b. 5
 c. 15 Correct Choice
 d. 25
 e. 75

$$\vec{v} = (-4 \sin t, 4 \cos t, -3) \quad |\vec{v}| = \sqrt{16 \sin^2 t + 16 \cos^2 t + 9} = 5$$

$$L = \int_{z=9}^{z=0} ds = \int_0^3 |\vec{v}| dt = \int_0^3 5 dt = [5t]_0^3 = 15$$

6. A box slides down the helical ramp $\vec{r}(t) = (4 \cos t, 4 \sin t, 9 - 3t)$ starting at height $z = 9$ and ending at height $z = 0$ under the action of the force $\vec{F} = (-yz, xz, 5z)$. Find the work done on the box.

- a. $\frac{9}{2}$
 b. 9
 c. $\frac{25}{2}$
 d. $\frac{27}{2}$ Correct Choice
 e. 27

$$\vec{v} = (-4 \sin t, 4 \cos t, -3) \quad \vec{F} = (-yz, xz, 5z) = (-4 \sin t(9 - 3t), 4 \cos t(9 - 3t), 5(9 - 3t))$$

$$\vec{F} \cdot \vec{v} = 16 \sin^2 t(9 - 3t) + 16 \cos^2 t(9 - 3t) - 15(9 - 3t) = 9 - 3t$$

$$W = \int_{z=9}^{z=0} \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F} \cdot \vec{v} dt = \int_0^3 (9 - 3t) dt = \left[9t - \frac{3}{2}t^2 \right]_0^3 = \frac{27}{2}$$

7. The diameter and height of a cylindrical trash can (no lid) are measured as $D = 30$ cm and $h = 40$ cm. The metal is 0.2 cm thick. Use differentials to estimate the volume of metal used to make the can.

- a. $165\pi \text{ cm}^3$
- b. $210\pi \text{ cm}^3$
- c. $285\pi \text{ cm}^3$ Correct Choice
- d. $330\pi \text{ cm}^3$
- e. $525\pi \text{ cm}^3$

$$V = \pi r^2 h \quad r = 15 \quad h = 40 \quad dr = 0.2 \quad dh = 0.2$$

$$dV = \frac{\partial V}{\partial r} dr + \frac{\partial V}{\partial h} dh = (2\pi rh)dr + (\pi r^2)dh = 2\pi \cdot 15 \cdot 40 \cdot 0.2 + \pi 15^2 \cdot 0.2 = 285\pi$$

$$V = \frac{1}{4}\pi D^2 h \quad D = 30 \quad h = 40 \quad dD = 0.4 \quad dh = 0.2$$

$$dV = \frac{\partial V}{\partial D} dD + \frac{\partial V}{\partial h} dh = \left(\frac{1}{2}\pi Dh\right)dD + \left(\frac{1}{4}\pi D^2\right)dh = \frac{1}{2}\pi \cdot 30 \cdot 40 \cdot 0.4 + \frac{1}{4}\pi 30^2 \cdot 0.2 = 285\pi$$

8. Find the equation of the plane tangent to the surface $z = x^3 y^2$ at the point $(2, 1)$. Then the z -intercept is $z =$

- a. -40
- b. 8
- c. -8
- d. 32
- e. -32 Correct Choice

$$f(x, y) = x^3 y^2 \quad f(2, 1) = 8 \quad \text{The tangent plane is}$$

$$f_x(x, y) = 3x^2 y^2 \quad f_x(2, 1) = 12 \quad z = f(2, 1) + f_x(2, 1)(x - 2) + f_y(2, 1)(y - 1)$$

$$f_y(x, y) = 2x^3 y \quad f_y(2, 1) = 16 \quad = 8 + 12(x - 2) + 16(y - 1)$$

$$= 12x + 15y - 32$$

The intercept is $z = -32$.

9. Find the equation of the plane tangent to the surface $12xyz - z^3 = 45$ at the point $(1, 2, 3)$. Then the z -intercept is $z =$

- a. 135
- b. 45
- c. $-\sqrt[3]{6}$
- d. -45 Correct Choice
- e. -135

$$F(x, y, z) = xyz - z^3 \quad \vec{\nabla}F = (12yz, 12xz, 12xy - 3z^2) \quad \vec{N} = \left. \vec{\nabla}F \right|_{(1,2,3)} = (72, 36, -3)$$

$$P = (1, 2, 3) \quad \vec{N} \cdot X = \vec{N} \cdot P \quad 72x + 36y - 3z = 72 \cdot 1 + 36 \cdot 2 - 3 \cdot 3 = 135$$

$$z\text{-intercept when: } x = 0, y = 0, \quad -3z = 135 \quad z = -45$$

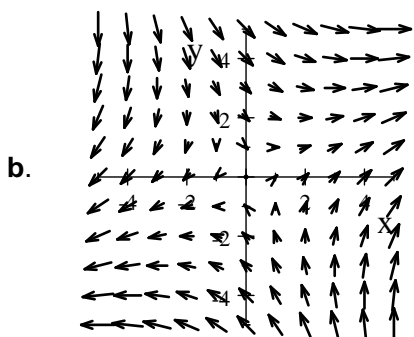
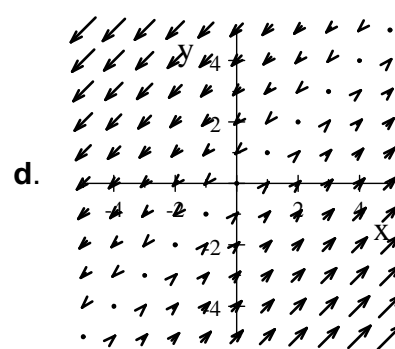
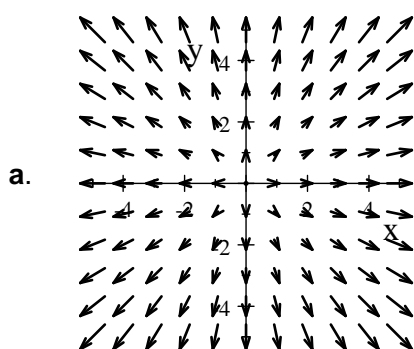
10. Starting from the point $(1, -2)$, find the maximum rate at which the function $f(x, y) = x^2y^3$ increases.

- a. 20 Correct Choice
- b. 25
- c. 400
- d. $(-16, 12)$
- e. $(16, -12)$

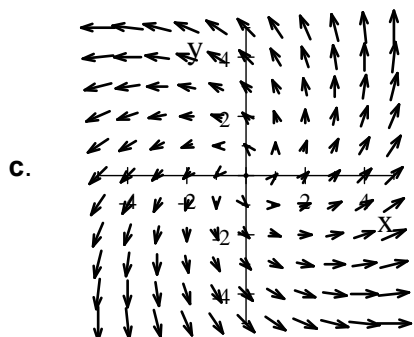
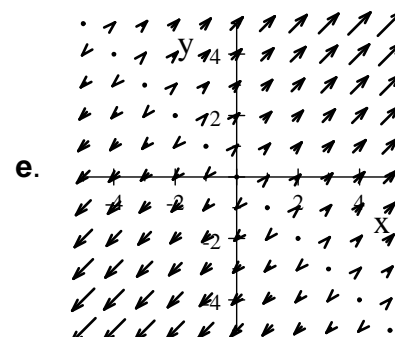
$$\vec{\nabla}f = (2xy^3, 3x^2y^2) \quad \vec{\nabla}f(1, -2) = (-16, 12)$$

$$\text{The maximum rate of increase is } |\vec{\nabla}f(1, -2)| = \sqrt{16^2 + 12^2} = \sqrt{256 + 144} = \sqrt{400} = 20$$

11. Which of the following is the plot of the vector field $F(x, y) = (x + y, x - y)$?



Correct Choice



When $y = x$, $F_2 = x - y = 0$ and \vec{F} is horizontal.
 When $y = -x$, $F_1 = x + y = 0$ and \vec{F} is vertical.
 So \vec{F} is (b).

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (20 points) Find the point on the curve $\vec{r}(t) = (e^t, \sqrt{2}t, e^{-t})$ where the curvature is a local maximum or local minimum. Is it a local maximum or local minimum?

HINTS: First find the curvature $\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$. Then find the critical point and apply the first or second derivative test.

$$\vec{v}(t) = (e^t, \sqrt{2}, -e^{-t}) \quad |\vec{v}| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t}$$

$$\vec{a}(t) = (e^t, 0, e^{-t})$$

$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ e^t & \sqrt{2} & -e^{-t} \\ e^t & 0 & e^{-t} \end{vmatrix} = \hat{i}(\sqrt{2}e^{-t}) - \hat{j}(1+1) + \hat{k}(-\sqrt{2}e^t) = (\sqrt{2}e^{-t}, -2, -\sqrt{2}e^t)$$

$$|\vec{v} \times \vec{a}| = \sqrt{2e^{-2t} + 4 + 2e^{2t}} = \sqrt{2} \sqrt{e^{-2t} + 2 + e^{2t}} = \sqrt{2}(e^t + e^{-t})$$

$$\kappa = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{2}(e^t + e^{-t})}{(e^t + e^{-t})^3} = \frac{\sqrt{2}}{(e^t + e^{-t})^2}$$

$$\kappa' = \frac{-2\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^3} = 0 \quad e^t = e^{-t} \quad e^{2t} = 1 \quad t = 0$$

$$\vec{r}(0) = (e^0, \sqrt{2} \cdot 0, e^{-0}) = \boxed{(1, 0, 1)}$$

$$\begin{aligned} \kappa'' &= \frac{-2\sqrt{2}(e^t + e^{-t})^3(e^t + e^{-t}) + 2\sqrt{2}(e^t - e^{-t})3(e^t + e^{-t})^2(e^t - e^{-t})}{(e^t + e^{-t})^6} \\ &= \frac{-2\sqrt{2}(e^t + e^{-t})^2 + 2\sqrt{2}(e^t - e^{-t})3(e^t - e^{-t})}{(e^t + e^{-t})^4} \end{aligned}$$

$$\kappa''(0) = \frac{-2\sqrt{2}(1+1)^2 + 2\sqrt{2}(1-1)3(1-1)}{(1+1)^4} = \frac{-2\sqrt{2}}{4} < 0$$

local maximum

13. (10 points) The pressure, P , density, D , and temperature, T , of a certain ideal gas are related by $P = 4DT$. A fly is currently at the point $\vec{r}(t_0) = (3, 2, 4)$ and has velocity $\vec{v}(t_0) = (2, 1, 2)$. At the point $(3, 2, 4)$, the density and temperature and their gradients are

$$D = 50 \quad \vec{\nabla}D = \left(\frac{\partial D}{\partial x}, \frac{\partial D}{\partial y}, \frac{\partial D}{\partial z} \right) = (0.1, 0.4, 0.2)$$

$$T = 300 \quad \vec{\nabla}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right) = (2, 3, 1)$$

Find the time rate of change of the pressure, $\frac{dP}{dt}$, as seen by the fly.

$$\begin{aligned} \frac{dP}{dt} &= \frac{dP}{dD} \frac{dD}{dt} + \frac{dP}{dT} \frac{dT}{dt} = \frac{dP}{dD} \left(\frac{dD}{dx} \frac{dx}{dt} + \frac{dD}{dy} \frac{dy}{dt} + \frac{dD}{dz} \frac{dz}{dt} \right) + \frac{dP}{dT} \left(\frac{dT}{dx} \frac{dx}{dt} + \frac{dT}{dy} \frac{dy}{dt} + \frac{dT}{dz} \frac{dz}{dt} \right) \\ &= 4T(\vec{\nabla}D \cdot \vec{v}) + 4D(\vec{\nabla}T \cdot \vec{v}) = 4 \cdot 300(0.2 + 0.4 + 0.4) + 4 \cdot 50(4 + 3 + 2) = 1200 + 1800 = 3000 \end{aligned}$$

14. (10 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\lim_{(x,y) \rightarrow (0,0)} \frac{3x^2y^2}{x^6 + 3y^3}$

Straight line approaches: $y = mx$

$$\lim_{\substack{y=mx \\ x \rightarrow 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x \rightarrow 0} \frac{3x^2m^2x^2}{x^6 + 3m^3x^3} = \lim_{x \rightarrow 0} \frac{3m^2x}{x^3 + 3m^3} = \frac{0}{3m^3} = 0$$

Quadratic approaches: $y = mx^2$

$$\lim_{\substack{y=mx^2 \\ x \rightarrow 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x \rightarrow 0} \frac{3x^2m^2x^4}{x^6 + 3m^3x^6} = \lim_{x \rightarrow 0} \frac{3m^2}{1 + 3m^3} \neq 0 \quad \text{if } m \neq 0.$$

Limit does not exist because these are different.

b. $\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2}$

Switch to polar: $x = r \cos \theta$ $y = r \sin \theta$

$$\lim_{(x,y) \rightarrow (0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{\substack{r \rightarrow 0 \\ \theta \text{ arbitrary}}} \frac{r \cos \theta r^2 \sin^2 \theta}{r^2} = \lim_{\substack{r \rightarrow 0 \\ \theta \text{ arbitrary}}} r \cos \theta \sin^2 \theta = 0$$

because $r \rightarrow 0$ while $\cos \theta \sin^2 \theta$ is bounded: $-1 \leq \cos \theta \sin^2 \theta \leq 1$.