Name________________________

1-11	/44	14	/10
12	/15	15	/20
13	/15	Total	/104

MATH 221 Final Fall 2009

Section 503 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Find the point where the line $x = 2t$, $y = 4 - t$, $z = -2 + t$ intersects the plane $x - y + z = -2$. At this point $x + y + z =$

a. -2

b. 0

c. 1

d. 4 Correct Choice

e. 6

$x - y + z = (2t) - (4 - t) + (-2 + t) = 4t - 6 = -2 \qquad t = 1$

$x = 2, \quad y = 3, \quad z = -1, \qquad x + y + z = 4$

2. Find the plane tangent to the graph of $z = x^2 y^3$ at the point $(2,1)$. The z-intercept is

a. -24

b. -16 Correct Choice

c. -4

d. 0

e. 4

$f(x,y) = x^2y^3 \qquad f_x(x,y) = 2xy^3 \qquad f_y(x,y) = 3x^2y^2$

$f(2,1) = 4 \qquad f_x(2,1) = 4 \qquad f_y(2,1) = 12$

Tan plane: $z = 4 + 4(x - 2) + 12(y - 1) = 4x + 12y - 16 \qquad z$-intercept $= -16$

3. Find the line perpendicular to the cone $z^2 - x^2 - y^2 = 0$ at the point $P = (4,3,5)$. This line intersects the xy-plane at

a. $(3,4,0)$

b. $(-4,-3,0)$

c. $(8,6,0)$ Correct Choice

d. $(-6,-8,0)$

e. $\left(\frac{4}{5},\frac{3}{5},0\right)$

$F = z^2 - x^2 - y^2 \qquad \vec{\nabla}F = (-2x,-2y,2z) \qquad \vec{N} = \vec{\nabla}F\Big|_{(4,3,5)} = (-8,-6,10)$

$X = P + t\vec{N} = (4,3,5) + t(-8,-6,10) = (4 - 8t, 3 - 6t, 5 + 10t)$

Intersects the xy-plane when $z = 0$ or $5 + 10t = 0$ or $t = -\frac{1}{2}$.

So $x = 4 - 8\left(-\frac{1}{2}\right) = 8$ and $y = 3 - 6\left(-\frac{1}{2}\right) = 6$

4. A box currently has length $L = 20$ cm which is increasing at 4 cm/sec, width $W = 15$ cm which is decreasing at 2 cm/sec, and height $H = 12$ cm which is increasing at 1 cm/sec. At what rate is the volume changing?

a. 3 cm^3/sec

b. 7 cm^3/sec

c. 540 cm^3/sec   Correct Choice

d. 1500 cm^3/sec

e. 3600 cm^3/sec

$$V = LWH \qquad \frac{dV}{dT} = WH\frac{dL}{dt} + LH\frac{dW}{dt} + LW\frac{dH}{dt} = 15 \cdot 12 \cdot 4 - 20 \cdot 12 \cdot 2 + 20 \cdot 15 \cdot 1 = 540$$

5. Duke Skywater is flying the Millenium Eagle through a galactic dust storm. Currently, his position is $P = (30, -20, 10)$ and his velocity is $\vec{v} = (-4, 3, 12)$. He measures that currently the dust density is $\rho = 450$ and its gradient is $\vec{\nabla}\rho = (2, -2, 1)$. Find the current rate of change of the dust density as seen by Duke.

a. -2 Correct Choice

b. 2

c. 110

d. 448

e. 560

$$\vec{\nabla}_{\vec{v}}\rho = \vec{v} \cdot \vec{\nabla}\rho = (-4, 3, 12) \cdot (2, -2, 1) = -8 - 6 + 12 = -2$$

6. Under the same conditions as in #5, in what **unit** vector direction should Duke travel to **decrease** the dust density as quickly as possible?

a. $(-2, 2, -1)$

b. $(2, -2, 1)$

c. $\left(\frac{4}{13}, \frac{-3}{13}, \frac{-12}{13}\right)$

d. $\left(\frac{-4}{13}, \frac{3}{13}, \frac{12}{13}\right)$

e. $\left(\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$ Correct Choice

$$\hat{u} = \frac{-\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \frac{-(2, -2, 1)}{\sqrt{4 + 4 + 1}} = \left(\frac{-2}{3}, \frac{2}{3}, \frac{-1}{3}\right)$$

7. The point $(2,-1)$ is a critical point of the function $f = \dfrac{x^2y^2 + 2x - 4y}{xy}$. Use the Second Derivative Test to classify the point.

- **a.** Local Minimum
- **b.** Local Maximum   Correct Choice
- **c.** Inflection Point
- **d.** Saddle Point
- **e.** Test Fails

$f = xy + \dfrac{2}{y} - \dfrac{4}{x} \qquad f_x = y + \dfrac{4}{x^2} \qquad f_y = x - \dfrac{2}{y^2}$

$f_{xx} = -\dfrac{8}{x^3} = -1 < 0 \qquad f_{yy} = \dfrac{4}{y^3} = -4 < 0 \qquad f_{xy} = 1 \qquad D = 4 - 1 = 3 > 0$ Maximum

8. Compute $\oint \vec{F} \cdot d\vec{s}$ counterclockwise around the rectangle $0 \le x \le \pi$ and $0 \le y \le 2\pi$ for $\vec{F} = (y\sin x + \sin y, x\cos y + \cos x)$.

HINT: Use the Fundamental Theorem of Calculus for Curves or Green's Theorem.

- **a.** -16π
- **b.** -8π Correct Choice
- **c.** 0
- **d.** 8π
- **e.** 16π

By Green's Theorem: $\vec{F} = (P,Q) \qquad P = y\sin x + \sin y \qquad Q = x\cos y + \cos x$

$$\oint \vec{F} \cdot d\vec{s} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint (\cos y - \sin x - \sin x - \cos y)\,dx\,dy = \int_0^{2\pi} \int_0^{\pi} -2\sin x\,dx\,dy$$

$$= 4\pi \Big[\cos x \Big]_0^{\pi} = -8\pi$$

9. Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the sphere $x^2 + y^2 + z^2 = 4$ with outward normal for $\vec{F} = (xy^2z, yz^2x, zx^2y)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

- **a.** 4π
- **b.** 12π
- **c.** $\dfrac{32}{3}\pi$
- **d.** $\dfrac{64}{3}\pi$
- **e.** 0 Correct Choice

By Stokes' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{s} = 0$ because there is no boundary curve.

By Gauss' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{\nabla} \times \vec{F}\,dV = 0$ because $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$.

10. Find the area of the piece of the paraboloid $z = x^2 + y^2$ above the circle $x^2 + y^2 \le 2$.

Note: The paraboloid may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2)$.

a. $-\frac{\pi}{32}\left[\ln\left(2\sqrt{2}+3\right) - 102\sqrt{2}\right]$

b. $\frac{\pi}{2}\left[\ln\left(2\sqrt{2}+3\right) + 6\sqrt{2}\right]$

c. $\frac{13\pi}{3}$ Correct Choice

d. $\frac{\pi}{6}(5^{3/2} - 1)$

e. $\frac{\pi}{6}(17^{3/2} - 1)$

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array}\left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & 2r) \\ (-r\sin\theta & r\cos\theta & 0) \end{array}\right|$$

$$\vec{N} = \hat{\imath}(-2r^2\cos\theta) - \hat{\jmath}(2r^2\sin\theta) + \hat{k}(r) = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{4r^4 + r^2} = r\sqrt{4r^2+1}$$

$$A = \iint 1\,dS = \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\,dr\,d\theta = \frac{2\pi}{12}(4r^2+1)^{3/2}\Big|_0^{\sqrt{2}} = \frac{\pi}{6}(9)^{3/2} - \frac{\pi}{6} = \frac{26\pi}{6} = \frac{13\pi}{3}$$

11. Use Gauss' Theorem to compute $\iint \vec{F}\cdot d\vec{S}$ outward through the complete surface of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$ for $\vec{F} = (x^2, xy, xz)$.

Note: The top of the tetrahedron is the plane $z = 6 - 3x - 2y$.

a. 4

b. 6

c. 9

d. 12 Correct Choice

e. 16

$$\vec{\nabla}\cdot\vec{F} = 2x + x + x = 4x$$

$$\iint_{\partial V}\vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \int_0^2\int_0^{3-\frac{3}{2}x}\int_0^{6-3x-2y} 4x\,dz\,dy\,dx = \int_0^2\int_0^{3-\frac{3}{2}x} 4x(6-3x-2y)\,dy\,dx$$

$$= \int_0^2 [24xy - 12x^2y - 4xy^2]_{y=0}^{3-\frac{3}{2}x}\,dx = \int_0^2\left[24x\left(3-\tfrac{3}{2}x\right) - 12x^2\left(3-\tfrac{3}{2}x\right) - 4x\left(3-\tfrac{3}{2}x\right)^2\right]dx$$

$$= \int_0^2\left[72x - 36x^2 - 36x^2 + 18x^3 - 4x\left(9 - 9x + \frac{9}{4}x^2\right)\right]dx$$

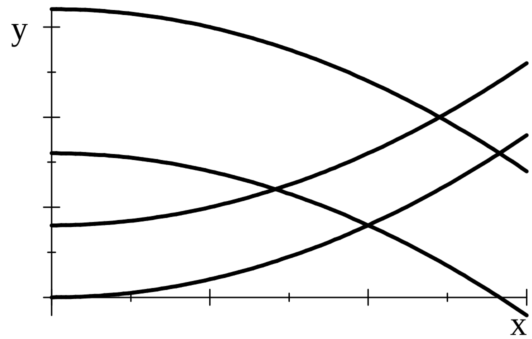
$$= \int_0^2 (36x - 36x^2 + 9x^3)\,dx = \left[18x^2 - 12x^3 + \frac{9}{4}x^4\right]_0^2 = 12$$

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (15 points) Compute $\iint_D x\,dx\,dy$ over the "diamond shaped" region D in the first quadrant bounded by the parabolas

$y = 16 - x^2$ $\quad$ $y = 8 - x^2$

$y = 4 + x^2$ $\quad$ and $\quad$ $y = x^2$



HINTS: Use the coordinates: $u = y + x^2$, $v = y - x^2$. Solve for x and y.

$$y = v + x^2 \qquad u = v + 2x^2 \qquad x^2 = \frac{u-v}{2} \qquad \boxed{x = \sqrt{\frac{u-v}{2}}} \qquad y = v + \frac{u-v}{2} \qquad \boxed{y = \frac{u+v}{2}}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\| \begin{matrix} \dfrac{1}{2\sqrt{\dfrac{u-v}{2}}} & \dfrac{1}{2} \\ \dfrac{1}{2\sqrt{\dfrac{u-v}{2}}} \dfrac{-1}{2} & \dfrac{1}{2} \end{matrix} \right\| = \left| \frac{1}{8\sqrt{\dfrac{u-v}{2}}} - -\frac{1}{8\sqrt{\dfrac{u-v}{2}}} \right| = \frac{1}{4\sqrt{\dfrac{u-v}{2}}}$$

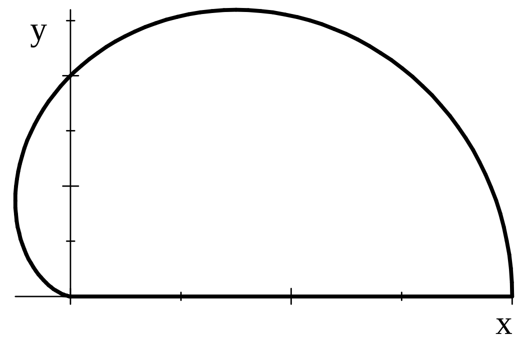
Integrand: $x = \sqrt{\dfrac{u-v}{2}}$ $\qquad$ Boundaries are:

$y = 16 - x^2 \Rightarrow y + x^2 = 16 \Rightarrow u = 16$ $\qquad$ $y = 8 - x^2 \Rightarrow y + x^2 = 8 \Rightarrow u = 8$

$y = 4 + x^2 \Rightarrow y - x^2 = 4 \Rightarrow v = 4$ $\qquad$ $y = x^2 \Rightarrow y - x^2 = 0 \Rightarrow v = 0$

$$\iint_R x\,dx\,dy = \int_0^4\int_8^{16} \sqrt{\frac{u-v}{2}}\,\frac{1}{4\sqrt{\dfrac{u-v}{2}}}\,du\,dv = \int_0^4\int_8^{16} \frac{1}{4}\,du\,dv = \frac{1}{4}[u]_8^{16}[v]_0^4 = \frac{1}{4}(8)(4) = 8$$

13. (15 points) Find the area and y-component of the centroid (center of mass with $\rho = 1$) of the upper half of the cardioid $r = 1 + \cos\theta$.



In polar coordinates: $0 \le r \le 1 + \cos\theta$ and $0 \le \theta \le \pi$.

$$A = \int_0^\pi \int_0^{1+\cos\theta} 1\, r\, dr\, d\theta = \int_0^\pi \left[\frac{r^2}{2} \right]_{r=0}^{1+\cos\theta} d\theta = \int_0^\pi \frac{(1+\cos\theta)^2}{2}\, d\theta$$

$$= \frac{1}{2} \int_0^\pi (1 + 2\cos\theta + \cos^2\theta)\, d\theta = \frac{1}{2}\left[\theta + 2\sin\theta + \left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4} \right) \right]_0^\pi$$

$$= \frac{1}{2}\left(\pi + \left(\frac{1}{2}\pi \right) \right) = \frac{3\pi}{4}$$

$$M_x = \int_0^\pi \int_0^{1+\cos\theta} r\sin\theta\, r\, dr\, d\theta = \int_0^\pi \left[\frac{r^3}{3} \right]_{r=0}^{1+\cos\theta} \sin\theta\, d\theta = \int_0^\pi \frac{(1+\cos\theta)^3}{3} \sin\theta\, d\theta$$

$$= -\int \frac{u^3}{3}\, du = -\frac{u^4}{12} = -\frac{(1+\cos\theta)^4}{12} \Bigg|_0^\pi = -0 + \frac{2^4}{12} = \frac{4}{3}$$

$$\bar{y} = \frac{M_x}{A} = \frac{4}{3}\frac{4}{3\pi} = \frac{16}{9\pi}$$

14. (10 points) Find 3 positive numbers x, y and z, whose sum is 120 such that $f(x,y,z) = xy^2z^3$ is a maximum.

METHOD 1: Lagrange Multipliers: $x + y + z = 120$

$f = xy^2z^3 \qquad \vec{\nabla} f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x + y + z \qquad \vec{\nabla} g = (1,1,1)$

$\vec{\nabla} f = \lambda \vec{\nabla} g \quad \Rightarrow \quad y^2z^3 = \lambda, \quad 2xyz^3 = \lambda, \quad 3xy^2z^2 = \lambda \quad \Rightarrow \quad y^2z^3 = 2xyz^3, \quad y^2z^3 = 3xy^2z^2$

$\Rightarrow \quad y = 2x, \quad z = 3x \quad \Rightarrow \quad x + y + z = x + 2x + 3x = 120 \quad \Rightarrow \quad 6x = 120 \quad \Rightarrow \quad x = 20$

$x = 20, \quad y = 40, \quad z = 60$

METHOD 2: Eliminate a Variable: $x + y + z = 120$

$x = 120 - y - z \quad \Rightarrow \quad f = (120 - y - z)y^2z^3 = 120y^2z^3 - y^3z^3 - y^2z^4$

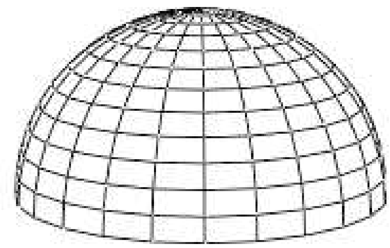
$$\begin{matrix} f_y = 240yz^3 - 3y^2z^3 - 2yz^4 = 0 \\ f_z = 360y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0 \end{matrix} \quad \Rightarrow \quad \begin{matrix} 240 - 3y - 2z = 0 \\ 360 - 3y - 4z = 0 \end{matrix}$$

Subtract: $120 - 2z = 0 \quad \Rightarrow \quad z = 60$

Substitute back: $240 - 3y - 120 = 0 \quad \Rightarrow \quad y = 40$

Substitute back: $x = 120 - y - z = 120 - 40 - 60 = 20$

15. (20 points) Verify Stokes' Theorem $\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial H} \vec{F} \cdot d\vec{s}$



for the vector field $\vec{F} = (y, -x, xz + yz)$

and the hemisphere $z = \sqrt{9 - x^2 - y^2}$ oriented up.

Use the following steps:

a. Parametrize the boundary curve and compute the line integral:

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$ oriented counterclockwise - OK

$\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 0)$

$$\oint_{\partial H} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -9\sin^2\theta - 9\cos^2\theta\, d\theta = -9\int_0^{2\pi} d\theta = -18\pi$$

b. Compute the surface integral using the parametrization:

$\vec{R}(\varphi, \theta) = (\quad 3\sin\varphi\cos\theta, \quad 3\sin\varphi\sin\theta, \quad 3\cos\varphi \quad)$

$\vec{e}_\varphi = (\quad 3\cos\varphi\cos\theta, \quad 3\cos\varphi\sin\theta, \quad -3\sin\varphi \quad)$

$\vec{e}_\theta = (\quad -3\sin\varphi\sin\theta, \quad 3\sin\varphi\cos\theta, \quad 0 \quad)$

$\vec{N} = \hat{\imath}(9\sin^2\varphi\cos\theta) - \hat{\jmath}(-9\sin^2\varphi\sin\theta) + \hat{k}(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta)$

$\quad = (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$ oriented up - OK

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & xz + yz \end{vmatrix} = \hat{\imath}(z) - \hat{\jmath}(z) + \hat{k}(-1 - 1) = (z, -z, -2) = (3\cos\varphi, -3\cos\varphi, -2)$$

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_H \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} (27\sin^2\varphi\cos\varphi\cos\theta - 27\sin^2\varphi\cos\varphi\sin\theta - 18\sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\, d\varphi\, d\theta \quad \text{because} \int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = 0$$

$$= 2\pi\int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\, d\varphi = -36\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = -18\pi$$