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|---|---|---|---|
| 1-12 | /60 | 14 | /15 |
| 13 | /15 | 15 | /15 |
| | | Total | /105 |

MATH 221 Exam 2 Spring 2011

Section 500 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

**1.** Compute $\int_0^2 \int_0^y xy\,dx\,dy$.

**a.** 1
**b.** 2 Correct Choice
**c.** 3
**d.** 4
**e.** $y^2$

$$\int_0^2 \int_0^y xy\,dx\,dy = \int_0^2 \frac{x^2 y}{2}\Big|_{x=0}^{y} dy = \frac{1}{2}\int_0^2 y^3\,dy = \frac{1}{2}\frac{y^4}{4}\Big|_{y=0}^{2} = 2$$

**2.** Find the area of one loop of the rose $r = \sin(3\theta)$.

**a.** $\frac{\pi}{12}$ Correct Choice
**b.** $\frac{\pi}{6}$
**c.** $\frac{\pi}{4}$
**d.** $\frac{\pi}{3}$
**e.** $\frac{\pi}{2}$

$\sin(3\theta) = 0$ at $\theta = 0$ and $3\theta = \pi$ or $\theta = \pi/3$

$$A = \int_0^{\pi/3} \int_0^{\sin(3\theta)} r\,dr\,d\theta = \int_0^{\pi/3} \left[\frac{r^2}{2}\right]_{r=0}^{\sin(3\theta)} d\theta = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \frac{1-\cos(6\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta - \frac{\sin(6\theta)}{6}\right]_{\theta=0}^{\pi/3} = \frac{\pi}{12}$$

**3.** Compute $\iiint x^2 + y^2\,dV$ over the region between the cones $z = \sqrt{x^2+y^2}$ and $z = 4 - \sqrt{x^2+y^2}$.

**a.** $\frac{8\pi}{3}$
**b.** $\frac{16\pi}{3}$
**c.** $\frac{32\pi}{3}$
**d.** $\frac{16\pi}{5}$
**e.** $\frac{32\pi}{5}$ Correct Choice

In cylindrical coordinates, the cones are $z = r$ and $z = 4 - r$ which intersect at $r = 2$.

$$\int_0^{2\pi}\int_0^2\int_r^{4-r} r^2\,r\,dz\,dr\,d\theta = 2\pi\int_0^2 \left[r^3 z\right]_{z=r}^{4-r} dr = 2\pi\int_0^2 r^3(4-2r)\,dr = 2\pi\left[r^4 - 2\frac{r^5}{5}\right]_{r=0}^{2} = 2\pi\left(16 - \frac{64}{5}\right) = \frac{32\pi}{5}$$

**4.** Find the mass of the hemisphere $x^2 + y^2 + z^2 \le 4$ with $y \ge 0$ if the density is $\delta = y$.

**a.** $\dfrac{\pi}{2}$

**b.** $\pi$

**c.** $2\pi$

**d.** $4\pi$ Correct Choice

**e.** $8$

$$M = \iiint \delta\, dV = \int_0^\pi \int_0^\pi \int_0^2 \rho \sin\varphi \sin\theta \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = \int_0^2 \rho^3 \, d\rho \int_0^\pi \sin\theta \, d\theta \int_0^\pi \sin^2\varphi \, d\varphi$$

$$= \left[ \frac{\rho^4}{4} \right]_0^2 \Big[ -\cos\theta \Big]_0^\pi \left[ \frac{1}{2} \left( \varphi - \frac{\sin 2\varphi}{2} \right) \right]_0^\pi = 4(2)\left( \frac{\pi}{2} \right) = 4\pi$$

**5.** Find the center of mass of the hemisphere $x^2 + y^2 + z^2 \le 4$ with $y \ge 0$ if the density is $\delta = y$.

**a.** $\left( 0, \dfrac{64\pi}{15}, 0 \right)$

**b.** $\left( 0, \dfrac{16}{15}, 0 \right)$ Correct Choice

**c.** $\left( 0, \dfrac{\pi^2}{12}, 0 \right)$

**d.** $\left( 0, \dfrac{15}{16}, 0 \right)$

**e.** $\left( 0, \dfrac{12}{\pi^2}, 0 \right)$

$$M_{xz} = \iiint y\delta \, dV = \int_0^\pi \int_0^\pi \int_0^2 \rho^2 \sin^2\varphi \sin^2\theta \rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi = \int_0^2 \rho^4 \, d\rho \int_0^\pi \sin^2\theta \, d\theta \int_0^\pi \sin^3\varphi \, d\varphi$$

$$= \left[ \frac{\rho^5}{5} \right]_0^2 \left[ \frac{1}{2} \left( \theta - \frac{\sin 2\theta}{2} \right) \right]_0^\pi \int_0^\pi (1 - \cos^2\varphi) \sin\varphi \, d\varphi = \frac{2^5}{5} \left( \frac{\pi}{2} \right) \left[ -\cos\varphi + \frac{\cos^3\varphi}{3} \right]_0^\pi$$

$$= \frac{16\pi}{5} \left( 1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{64\pi}{15}$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{64\pi}{15} \frac{1}{4\pi} = \frac{16}{15} \qquad \bar{x} = \bar{z} = 0 \text{ by symmetry.}$$

**6.** Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-16x^2 y, 9xy^2)$ counterclockwise around the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

HINTS: The ellipse may be parametrized by $\vec{r}(\theta) = (3\cos\theta, 4\sin\theta)$.

Since $\sin(2\theta) = 2\sin\theta\cos\theta$, we have $4\sin^2\theta\cos^2\theta = \sin^2(2\theta)$.

**a.** $-864\pi$

**b.** $-288\pi$

**c.** $144\pi$

**d.** $288\pi$

**e.** $864\pi$ Correct Choice

$$\vec{F}(\vec{r}(\theta)) = (-16 \cdot 9\cos^2\theta \cdot 4\sin\theta, 9 \cdot 3\cos\theta \cdot 16\sin^2\theta) \qquad \vec{v} = (-3\sin\theta, 4\cos\theta)$$

$$\vec{F} \cdot \vec{v} = 64 \cdot 27\cos^2\theta\sin^2\theta + 27 \cdot 64\cos^2\theta\sin^2\theta = 27 \cdot 64 \cdot 2\cos^2\theta\sin^2\theta = 27 \cdot 32\sin^2(2\theta)$$

$$= 27 \cdot 32 \frac{1 - \cos(4\theta)}{2} = 27 \cdot 16(1 - \cos(4\theta))$$

$$\oint \vec{F} \cdot d\vec{s} = \oint \vec{F} \cdot \vec{v} \, d\theta = \int_0^{2\pi} 27 \cdot 16(1 - \cos(4\theta)) \, d\theta = 27 \cdot 16 \left[ \theta - \frac{\sin 4\theta}{4} \right]_0^{2\pi} = 864\pi$$

**7**. The point $\left(\frac{\pi}{3}, \frac{\pi}{6}\right)$ is a critical point of the function $f(x,y) = \sin(x)\cos(y) - \frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4}y$
Use the Second Derivative Test to classify this critical point.

- **a**. Local Maximum      Correct Choice
- **b**. Local Minimum
- **c**. Inflection Point
- **d**. Saddle Point
- **e**. Test Fails

$$f_x = \cos(x)\cos(y) - \frac{\sqrt{3}}{4} \qquad f_x\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4} = 0$$

$$f_y = -\sin(x)\sin(y) + \frac{\sqrt{3}}{4} \qquad f_y\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{4} = 0$$

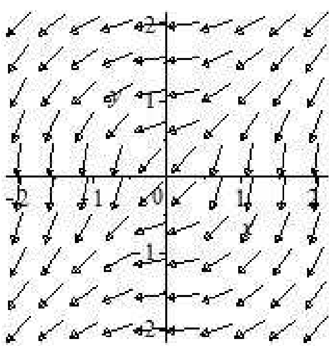
$$f_{xx} = -\sin(x)\cos(y) \qquad f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = -\frac{3}{4} < 0$$

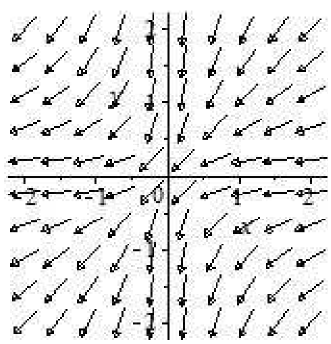
$$f_{yy} = -\sin(x)\cos(y) \qquad f_{yy}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = -\frac{3}{4}$$

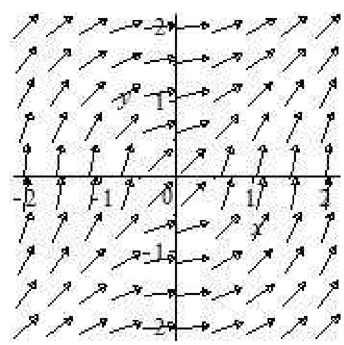
$$f_{xy} = -\cos(x)\sin(y) \qquad f_{xy}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = -\frac{1}{4}$$

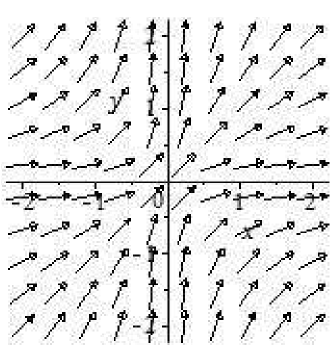
$$D = f_{xx}f_{yy} - f_{xy}^2 \qquad D\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{1}{2} > 0 \quad \text{Local Maximum}$$

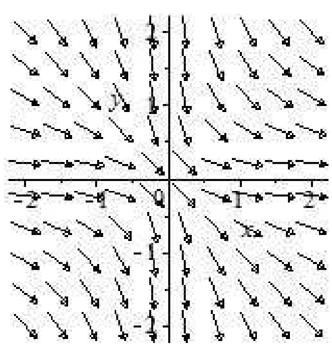
**8**. Which of the following is the plot of the vector field $\vec{F}(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left( |x|,\ |y| \right)$ ?

a. 

b. 

c. 

d. 

e. 

(d) Correct Choice

All arrows must be up and right. So (c) or (d). On the $x$-axis, $y = 0$ and so the arrows are horizontal there. So (d).

**9**. Find the area of the parameric surface $\vec{R}(u,t) = \left(ue^t, ue^{-t}, \sqrt{2}\,u\right)$ for $0 \le u \le 2$ and $0 \le t \le 1$.
HINT: Look for a perfect square.

**a**. $2\sqrt{2}\left(e + \frac{1}{e} - 2\right)$
**b**. $2\sqrt{2}\left(e + \frac{1}{e}\right)$
**c**. $2\sqrt{2}\left(e - \frac{1}{e} - 2\right)$
**d**. $2\sqrt{2}\left(e - \frac{1}{e}\right)$ Correct Choice
**e**. $2\left(e + \frac{1}{e}\right)$

$$\begin{array}{rl} & \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (e^t & e^{-t} & \sqrt{2}) \\ (ue^t & -ue^{-t} & 0) \end{array} \right| \end{array} \quad \vec{e}_u = (e^t, e^{-t}, \sqrt{2}), \quad \vec{e}_t = (ue^t, -ue^{-t}, 0)$$

$$\vec{N} = \vec{e}_u \times \vec{e}_t = \hat{\imath}\left(\sqrt{2}\,ue^{-t}\right) - \hat{\jmath}\left(-\sqrt{2}\,ue^t\right) + \hat{k}(-u-u) = \left(\sqrt{2}\,ue^{-t}, \sqrt{2}\,ue^t, -2u\right)$$

$$\left|\vec{N}\right| = \sqrt{2u^2e^{-2t} + 2u^2e^{2t} + 4u^2} = u\sqrt{2}\sqrt{e^{-2t} + 2 + e^{2t}} = u\sqrt{2}\,(e^{-t} + e^t)$$

$$A = \iint dS = \iint \left|\vec{N}\right| du\,dt = \int_0^1 \int_0^2 u\sqrt{2}\,(e^{-t} + e^t)\,du\,dt = \sqrt{2}\left[\frac{u^2}{2}\right]_0^2 \left[-e^{-t} + e^t\right]_0^1 = 2\sqrt{2}\left(e - \frac{1}{e}\right)$$

**10**. Find the equation of the plane tangent to the parameric surface $\vec{R}(u,t) = \left(ue^t, ue^{-t}, \sqrt{2}\,u\right)$ at the point $P = \vec{R}(2,0)$ where $u = 2$ and $t = 0$.
Hint: Evaluate the normal $\vec{N}$ at $u = 2$ and $t = 0$.

**a**. $x + y - \sqrt{2}\,z = -4\sqrt{2}$
**b**. $x + y - \sqrt{2}\,z = 0$ Correct Choice
**c**. $x + y - \sqrt{2}\,z = 16\sqrt{2}$
**d**. $\sqrt{2}\,x - \sqrt{2}\,y + 2z = -4\sqrt{2}$
**e**. $\sqrt{2}\,x - \sqrt{2}\,y + 2z = 0$

$P = \vec{R}(2,0) = \left(2, 2, 2\sqrt{2}\right)$ $\quad \vec{N} = \left(\sqrt{2}\,ue^{-t}, \sqrt{2}\,ue^t, -2u\right) = \left(2\sqrt{2}, 2\sqrt{2}, -4\right)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad 2\sqrt{2}\,x + 2\sqrt{2}\,y - 4z = 2\sqrt{2}\,(2) + 2\sqrt{2}\,(2) - 4\left(2\sqrt{2}\right) = 0$

**11**. If $\vec{F} = (xy\tan z, yz\cos x, xz\sin y)$, then $\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} =$

- **a**. $-2z\cos y - 2x(\tan^2 z + 1)$
- **b**. $-2z\cos y + 2x(\tan^2 z + 1)$
- **c**. $2z\cos y - 2x(\tan^2 z + 1)$
- **d**. $2z\cos y + 2x(\tan^2 z + 1)$
- **e**. $0$ Correct Choice

$\vec{\nabla}\cdot\vec{\nabla}\times\vec{F} = 0$ for any twice differentiable vector field.

**12**. Let $f$ be the scalar potential for $\vec{F} = (2xz - 3y, 8yz - 3x, x^2 + 4y^2 + 2z)$ for which $f(0,0,0) = 0$. Then $f(1,1,1) =$

- **a**. $1$
- **b**. $2$
- **c**. $3$ Correct Choice
- **d**. $4$
- **e**. $5$

$\vec{\nabla}f = \vec{F}$ or (1) $\partial_x f = 2xz - 3y$ (2) $\partial_y f = 8yz - 3x$ (3) $\partial_z f = x^2 + 4y^2 + 2z$

(1) $\Rightarrow$ $f = x^2 z - 3xy + g(y,z)$ $\Rightarrow$ (4) $\partial_y f = -3x + \partial_y g$

(2) and (4) $\Rightarrow$ $\partial_y g = 8yz$ $\Rightarrow$ $g = 4y^2 z + h(z)$ $\Rightarrow$ $f = x^2 z - 3xy + 4y^2 z + h(z)$

$\Rightarrow$ (5) $\partial_z f = x^2 + 4y^2 + \dfrac{dh(z)}{dz}$

(3) and (5) $\Rightarrow$ $\dfrac{dh(z)}{dz} = 2z$ $\Rightarrow$ $h = z^2 + C$ $\Rightarrow$ $f = x^2 z - 3xy + 4y^2 z + z^2 + C$

To have $f(0,0,0) = 0$ $\Rightarrow$ $C = 0$. So $f(1,1,1) = 1 - 3 + 4 + 1 = 3$.

Work Out: (Points indicated. Part credit possible. Show all work.)

**13**. (15 points) The plane $x + 2y + 4z = 8$ intersects the 1st octant $(x > 0, y > 0, z > 0)$ in a triangle. Find the point on this triangle at which the function $f = xy^2z^3$ is a maximum.

**Method of Eliminating a Variable**:

$x = 8 - 2y - 4z \qquad f = (8 - 2y - 4z)y^2z^3 = 8y^2z^3 - 2y^3z^3 - 4y^2z^4$

$f_y = 16yz^3 - 6y^2z^3 - 8yz^4 = 0 \qquad \Rightarrow \qquad 2yz^3(8 - 3y - 4z) = 0$

$f_z = 24y^2z^2 - 6y^3z^2 - 16y^2z^3 = 0 \qquad \Rightarrow \qquad 2y^2z^2(12 - 3y - 8z) = 0$

Since $f = xy^2z^3$ is positive in the 1st octant, we know the solution cannot have $y = 0$ or $z = 0$.

So we solve $8 - 3y - 4z = 0$ and $12 - 3y - 8z = 0$.

Solving each for $3y$ and equating gives $3y = 8 - 4z = 12 - 8z$, or $4z = 4$. So $z = 1$.

Then $3y = 4$. So $y = \frac{4}{3}$, and $x = 8 - 2\left(\frac{4}{3}\right) - 4(1) = \frac{4}{3}$.

The point is $\left(\frac{4}{3}, \frac{4}{3}, 1\right)$.

**Method of Lagrange Multipliers**:

$f = xy^2z^3 \qquad \vec{\nabla} f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x + 2y + 4z \qquad \vec{\nabla} g = (1, 2, 4)$

Lagrange equations: $y^2z^3 = \lambda$, $2xyz^3 = 2\lambda$, $3xy^2z^2 = 4\lambda$

Substitute the first eq into the other two: $2xyz^3 = 2y^2z^3$, $3xy^2z^2 = 4y^2z^3$.

Since $f = xy^2z^3$ is positive in the 1st octant, we know the solution cannot have $x = 0$ or $y = 0$ or $z = 0$. So we cancel: $x = y$, $3x = 4z$, So $y = x$ and $z = \frac{3}{4}x$.

Substitute into the constraint: $x + 2y + 4z = 8 \quad x + 2x + 3x = 8 \quad 6x = 8 \quad x = \frac{4}{3} \quad y = \frac{4}{3} \quad z = 1$.

The point is $\left(\frac{4}{3}, \frac{4}{3}, 1\right)$.

**14**. (15 points) Compute $\iiint \vec{\nabla} \cdot \vec{F}\, dV$ for $\vec{F} = (xy^2, yz^2, zx^2)$ over the solid above the cone $z = \sqrt{x^2 + y^2}$ below the sphere $x^2 + y^2 + z^2 = 4$.

$\vec{\nabla} \cdot \vec{F} = y^2 + z^2 + x^2 = \rho^2$ in spherical coordinates and $dV = \rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi$

The cone is $\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta} = \rho\sin\varphi$. So $\tan\varphi = 1$ or $\varphi = \pi/4$.

$$\iiint \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^2 \rho^2\rho^2 \sin\varphi\, d\rho\, d\theta\, d\varphi = \left[\frac{\rho^5}{5}\right]_0^2 (2\pi)\left[-\cos\varphi\right]_0^{\pi/4} = \frac{64\pi}{5}\left(1 - \frac{1}{\sqrt{2}}\right)$$

**15**. (15 points) Compute $\iint_E \vec{\nabla}\times\vec{F}\cdot\hat{k}\,dx\,dy$ for $\vec{F} = (-16x^2y, 9xy^2, 0)$ over the interior of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

HINTS: First compute $\vec{\nabla}\times\vec{F}\cdot\hat{k}$ in rectangular coordinates.

Then compute the integral in elliptic coordinates $x = 3u\cos\theta$, $y = 4u\sin\theta$.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -16x^2y & 9xy^2 & 0 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(9y^2 - -16x^2)$$

$$\vec{\nabla}\times\vec{F}\cdot\hat{k} = 9y^2 + 16x^2 = 9(4u\sin\theta)^2 + 16(3u\cos\theta)^2 = 9\cdot 16u^2 = 144u^2$$

$$\frac{\partial(x,y)}{\partial(u,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 3\cos\theta & 4\sin\theta \\ -3u\sin\theta & 4u\cos\theta \end{vmatrix} = 12u \qquad dx\,dy = J\,du\,d\theta = 12u\,du\,d\theta$$

$$\iint_E \vec{\nabla}\times\vec{F}\cdot\hat{k}\,dx\,dy = \int_0^{2\pi}\int_0^1 144u^2\,12u\,du\,d\theta = 2\pi 144\cdot 12\left[\frac{u^4}{4}\right]_0^1 = 864\pi$$