Multiple Choice: (5 points each. No part credit.)

1. Find the area of the triangle whose vertices are
   \( P = (2, 4, -3), \quad Q = (3, 4, -2) \quad \text{and} \quad R = (0, 6, -3). \)
   a. 12
   b. 6
   c. \( 2\sqrt{3} \)
   d. \( \sqrt{3} \) Correct Choice
   e. 1

SOLUTION:
   \[
   \overrightarrow{PQ} = Q - P = (1, 0, 1) \quad \overrightarrow{PR} = R - P = (-2, 2, 0)
   \]
   \[
   \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ -2 & 2 & 0 \end{vmatrix} = \hat{i}(0 - 2) - \hat{j}(0 - 2) + \hat{k}(2 - 0) = \langle -2, -2, 2 \rangle
   \]
   \[
   A = \frac{1}{2} |\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2} \sqrt{4 + 4 + 4} = \sqrt{3}
   \]

2. Which of the following is the hyperplane in \( \mathbb{R}^4 \) which contains the point \( P = (x, y, z, w) = (8, 4, 2, 1) \)
   and is tangent to the vectors \( \vec{p} = \langle 2, 1, 0, 0 \rangle \quad \vec{q} = \langle 0, 2, 1, 0 \rangle \) and \( \vec{r} = \langle 0, 0, 2, 1 \rangle \)?
   HINT: What is the vector perpendicular to the hyperplane?
   a. \( 8x + 4y + 2z + w = 85 \)
   b. \( 8x - 4y + 2z - w = 51 \)
   c. \( x + 2y + 4z + 8w = 32 \)
   d. \( x - 2y + 4z - 8w = 0 \) Correct Choice
   e. \( x - y + z - w = 5 \)

SOLUTION:
   \[
   \vec{N} = \perp (\vec{p}, \vec{q}, \vec{r}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{vmatrix}
   \]
   \[
   \vec{N} = \vec{i}(1 - 0) - \vec{j}(2 - 0) + \vec{k}(4 - 0) - \hat{i}(8 - 0) = \langle 1, 2, 4, 8 \rangle
   \]
   \[
   \vec{N} \cdot X = \vec{N} \cdot P = x - 2y + 4z - 8w = 1 \cdot 8 - 2 \cdot 4 + 4 \cdot 2 - 8 \cdot 1 = 0
   \]
3. The quadratic surface \( x^2 + y^2 - z^2 + 4x + 4y - 6z = 0 \) is
   a. a cone.
   b. an elliptic hyperboloid
   c. a hyperbolic paraboloid
   d. a hyperboloid of 1 sheet  
   e. a hyperboloid of 2 sheets  Correct Choice

SOLUTION: We complete the squares to get 
\[(x + 2)^2 + (y + 2)^2 - (z + 3)^2 = 4 + 4 - 9 = -1\]
Since there are squares on each variable with different signs and non-zero on the right, it is a hyperboloid. Rearranging, we have 
\[(z + 3)^2 = 1 + (x + 2)^2 + (y + 2)^2\]
So \((z + 3)^2\) can never be zero and it must be a hyperboloid of 2 sheets.

4. The plot at the right is the graph of which equation?
   a. \( z = x^2 - y^2 \)
   b. \( z = -x^2 + y^2 \)  Correct Choice
   c. \( z^2 = x^2 - y^2 \)
   d. \( z^2 = -x^2 + y^2 \)
   e. \( z^2 - x^2 - y^2 = 1 \)

SOLUTION: The saddle surface is a hyperbolic paraboloid: (a) or (b).
Since it goes down in the \(x\)-direction and up in the \(y\)-direction, it is (a).

5. An airplane is travelling due South with constant speed and constant altitude as it flies over College Station. Since its path is part of a circle around the earth, its acceleration points directly toward the center of the earth. In which direction does it binormal \( \vec{B} \) point?
   a. Up
   b. North
   c. South
   d. East  Correct Choice
   e. West

SOLUTION: \( \vec{v} \) is South. \( \vec{a} \) is Down.
So \( \vec{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} \) points East by the right hand rule.
6. For the curve \( \vec{r}(t) = (4 \cos t, \ 3t, \ 4 \sin t) \) which of the following is FALSE?

   a. \( \vec{v} = \langle -4 \sin t, \ 3, \ 4 \cos t \rangle \)
   
   b. \( \vec{a} = \langle -4 \cos t, \ 0, \ -4 \sin t \rangle \)
   
   c. \( |\vec{v}| = 25 \) Correct Choice
   
   d. Arc length between \( t = 0 \) and \( t = 2\pi \) is \( 10\pi \)
   
   e. \( a_T = 0 \)

SOLUTION: \( \vec{v} \) and \( \vec{a} \) are correct by differentiation.

\[
|\vec{v}| = \sqrt{9 + 16 \sin^2 t + 16 \cos^2 t} = 5 \quad a_T = \frac{d|\vec{v}|}{dt} = 0
\]

\[
L = \int ds = \int |\vec{v}| dt = \int_{0}^{2\pi} 5 dt = [5t]_{0}^{2\pi} = 10\pi
\]

7. A wire in the shape of the curve \( \vec{r}(t) = (4 \cos t, \ 3t, \ 4 \sin t) \) has linear mass density \( \rho = y + z \). Find its total mass between \( t = 0 \) and \( t = 2\pi \).

   a. \( 30\pi^2 \) Correct Choice
   
   b. \( 6\pi^2 \)
   
   c. \( 30\pi \)
   
   d. \( 12\pi \)
   
   e. \( 6\pi \)

SOLUTION: \( M = \int \rho ds = \int (y + z)|\vec{v}| dt = \int_{0}^{2\pi} (3t + 4 \sin t) 5 dt = \left[ \frac{15t^2}{2} - 20 \cos t \right]_{0}^{2\pi} = 30\pi^2 \)

8. Find the work done to move an object along the curve \( \vec{r}(t) = (4 \cos t, \ 3t, \ 4 \sin t) \) between \( t = 0 \) and \( t = 2\pi \) by the force \( \vec{F} = \langle z, 0, -x \rangle \).

   a. \( 32\pi \)
   
   b. \( 25\pi \)
   
   c. \( -25\pi^2 \)
   
   d. \( -25\pi \)
   
   e. \( -32\pi \) Correct Choice

SOLUTION: \( \vec{F}(\vec{r}(t)) = \langle 4 \sin t, \ 0, \ -4 \cos t \rangle \quad \vec{v} = \langle -4 \sin t, \ 3, \ 4 \cos t \rangle \)

\[
W = \int F \cdot ds = \int_{0}^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v} dt = \int_{0}^{2\pi} (-16 \sin^2 t - 16 \cos^2 t) dt = \int_{0}^{2\pi} -16 dt = -32\pi
\]
9. Find the plane tangent to the graph of \( z = xe^y \) at the point \( (2, 0) \). Its \( z \)-intercept is

a. \(-e\)
b. \(-2\)
c. 0 \hspace{1cm} \text{Correct Choice}
d. 2
e. \(e\)

SOLUTION:
\[
f = xe^y \hspace{1cm} f(2, 0) = 2 \hspace{1cm} z = f(2, 0) + f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0)
\]
\[
f_x = e^y \hspace{1cm} f_x(2, 0) = 1 \hspace{1cm} = 2 + 1(x - 2) + 2y
\]
\[
f_y = xe^y \hspace{1cm} f_y(2, 0) = 2 \hspace{1cm} \text{When } x = y = 0, \text{ we have } z = 2 + (-2) = 0.
\]

10. Find the plane tangent to the graph of \( xz^3 + zy^2 + yx^4 = 42 \) at the point \( (1, 2, 0) \). Its \( z \)-intercept is

a. \(\frac{2}{5}\)
b. \(\frac{4}{5}\)
c. \(\frac{5}{4}\)
d. \(\frac{5}{2}\) \hspace{1cm} \text{Correct Choice}
e. 10

SOLUTION: \( F(x, y, z) = xz^3 + zy^2 + yx^4 \hspace{1cm} \vec{V}F = \langle z^3 + 4yx^3, 2zy + x^4, 3xz^2 + y^2 \rangle \)
\[
\vec{N} = \left| \vec{V}F \right|_{(1,2,0)} = \langle 8, 1, 4 \rangle \hspace{1cm} \vec{N} \cdot \vec{X} = \vec{N} \cdot \vec{P} \hspace{1cm} 8x + y + 4z = 8 \cdot 1 + 2 + 4 \cdot 0 = 10
\]
When \( x = y = 0 \), we have \( z = \frac{5}{2} \).
11. Hans Duo is currently at \((x, y, z) = (3, 2, 1)\) and flying the Milenium Eagle through a deadly polaron field whose density is \(\rho = x^2z + yz^2\). In what unit vector direction should he travel to **reduce** the density as fast as possible?

a. \((-6, -1, -13)\)  

b. \(\frac{1}{\sqrt{206}}(-6, -1, -13)\)  **Correct Choice**  

c. \((6, 1, 13)\)  

d. \(\frac{1}{\sqrt{206}}(-6, -1, -13)\)  

e. \(\frac{1}{\sqrt{206}}(6, -1, 13)\)

**SOLUTION:**  
\[ \vec{v}_n = (2xz, x^2, x^2 + 2yz) \]  
\[ \vec{v} = \left. \vec{v}_n \right|_{(3, 2, 1)} = (-6, -1, -13) \]  
\[ |\vec{v}| = \sqrt{36 + 1 + 169} = \sqrt{206} \]  
\[ \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{206}}(-6, -1, -13) \]

12. The point \((x, y) = (9, 3)\) is a critical point of the function \(f(x, y) = x^2 - 2xy^2 + 4y^3\). Use the Second Derivative Test to classify this critical point.

a. local minimum  

b. local maximum  

c. saddle point  **Correct Choice**  

d. TEST FAILS

**SOLUTION:**  
\[ f_x = 2x - 2y^2 \quad \Rightarrow \quad f_x(9, 3) = 18 - 18 = 0 \quad \text{Checked} \]  
\[ f_y = -4xy + 12y^2 \quad \Rightarrow \quad f_y(9, 3) = -4 \cdot 27 + 12 \cdot 9 = 0 \quad \text{Checked} \]  
\[ f_{xx} = 2 \quad \Rightarrow \quad f_{xx}(9, 3) = 2 \]  
\[ f_{yy} = -4x + 24y \quad \Rightarrow \quad f_{yy}(9, 3) = -36 + 72 = 36 \]  
\[ f_{xy} = -4y \quad \Rightarrow \quad f_{xy}(9, 3) = -12 \]  
\[ D = f_{xx}f_{yy} - f_{xy}^2 = 2 \cdot 36 - 12^2 = -72 \]  
Since \(D < 0\) it is a saddle point.
13. Find the scalar and vector projections of the vector \( \vec{a} = (2, -1, 2) \) along the vector \( \vec{b} = (1, 2, -2) \).

**SOLUTION:**

\[
\vec{a} \cdot \vec{b} = 2 - 2 - 4 = -4 \\
\vec{b} \cdot \vec{b} = 4 + 1 + 4 = 9 \\
|\vec{b}| = 3
\]

\[
\text{comp}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{-4}{3}
\]

\[
\text{proj}_{\vec{b}} \vec{a} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{-4}{9} (1, 2, -2) = \left\langle \frac{-4}{9}, \frac{-8}{9}, \frac{8}{9} \right\rangle
\]

14. The pressure, \( P \), volume, \( V \), and temperature, \( T \), of an ideal gas are related by

\[
P = \frac{kT}{V}
\]

for some constant \( k \). For a certain sample \( k = 10 \text{ cm}^3\text{-atm} \text{ °K}^{-1} \).

At a certain instant, the volume and temperature are \( V = 2000 \text{ cm}^3 \) and \( T = 300 \text{ °K} \), and are increasing at \( \frac{dV}{dt} = 40 \text{ cm}^3/\text{sec} \), and \( \frac{dT}{dt} = 5 \text{ °K}/\text{sec} \).

At that instant, what is the pressure, is it increasing or decreasing and at what rate?

**SOLUTION:**

\[
P = \frac{kT}{V} = \frac{10 \cdot 300}{2000} \text{ cm}^3\text{-atm} \text{ °K}^{-1} \text{ cm}^3 = 1.5 \text{ atm}
\]

\[
\frac{dP}{dt} = \frac{\partial P}{\partial V} \frac{dV}{dt} + \frac{\partial P}{\partial T} \frac{dT}{dt} = -\frac{kT}{V^2} \frac{dV}{dt} + \frac{k}{V} \frac{dT}{dt} = -\frac{10 \cdot 300}{2000^2} \cdot 40 + \frac{10}{2000} \cdot 5 = -\frac{5}{1000} = -0.005 \text{ atm/sec}
\]

Since \( \frac{dP}{dt} \) is negative, the pressure is decreasing.
15. If two resistors, with resistances $R_1$ and $R_2$, are arranged in parallel, the total resistance $R$ is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} \quad \text{or} \quad R = \frac{R_1 R_2}{R_1 + R_2}$$

If $R_1 = 4\Omega$ and $R_2 = 6\Omega$, and the uncertainty in the measurement of $R_1$ is $\Delta R_1 = 0.03\Omega$ and for $R_2$ is $\Delta R_2 = 0.02\Omega$. Find $R$ and use differentials to estimate the uncertainty in the measurement of $R$.

**SOLUTION:** $R = \frac{4 \cdot 6}{4 + 6} = 2.4\Omega$.

$$\Delta R = \frac{\partial R}{\partial R_1} \Delta R_1 + \frac{\partial R}{\partial R_2} \Delta R_2 = (R_1 + R_2 - R_1 R_2(1) \Delta R_1 + \frac{(R_1 + R_2) R_1 - R_1 R_2(1) \Delta R_2}{(R_1 + R_2)^2}$$

$$= \frac{(R_2)^2}{(R_1 + R_2)^2} \Delta R_1 + \frac{(R_1)^2}{(R_1 + R_2)^2} \Delta R_2 = \frac{6^2}{(4 + 6)^2} \cdot 0.03 + \frac{4^2}{(4 + 6)^2} \cdot 0.02 = \frac{1.08 + 0.32}{100} = 0.014\Omega$$

16. Find the point(s) on the surface $z^2 = 46 - 4x - 2y$ which are closest to the origin.

**HINT:** Explain why you can minimize the square of the distance instead of the distance. Use the Second Derivative Test to check it is a local minimum.

**SOLUTION:** We need to minimize the distance from the point $(x, y, \pm \sqrt{46 - 4x - 2y})$ to the origin. We can minimize the square of the distance because as the distance decreases, so does its square. So we minimize $f = x^2 + y^2 + z^2 = x^2 + y^2 + 46 - 4x - 2y$

$f_x = 2x - 4 = 0 \quad \Rightarrow \quad x = 2$

$f_y = 2y - 2 = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow \quad z = \pm \sqrt{46 - 4x - 2y} = \pm \sqrt{36} = \pm 6$

So the points are $(2, 1, 6)$ and $(2, 1, -6)$

$f_{xx} = 2 > 0 \quad f_{yy} = 2 \quad f_{xy} = 0 \quad D = f_{xx} f_{yy} - f_{xy}^2 = 4 > 0 \quad \text{local minimum}$