1-8	/48
9	/12
10	/20
11	/20
Total	/100

Name__________________________

MATH 221 Exam 2 Fall 2012

Sections 503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Compute $\int_0^2 \int_x^2 5y^3\, dy\, dx$.

a. 68

b. 48

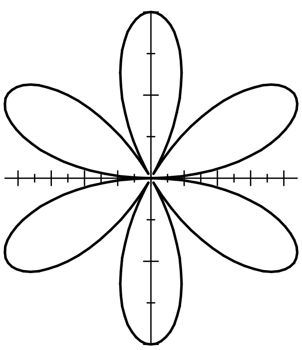
c. 32 Correct Choice

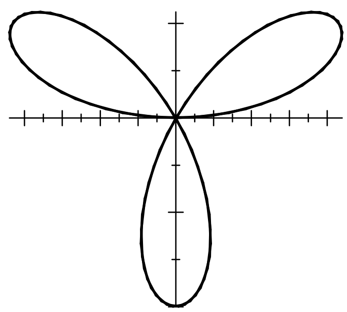
d. 27

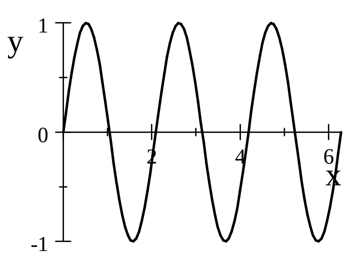
e. 15

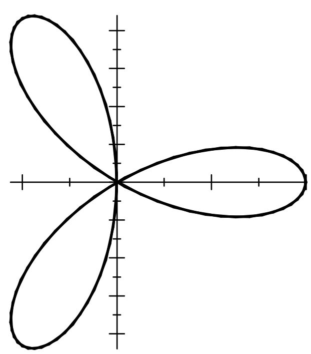
SOLUTION: $\int_0^2 \int_x^2 5y^3\, dy\, dx = \int_0^2 \left[5\frac{y^4}{4} \right]_{y=x}^2 dx = \int_0^2 20 - 5\frac{x^4}{4}\, dx = \left[20x - \frac{x^5}{4} \right]_0^2 = 40 - 8 = 32$

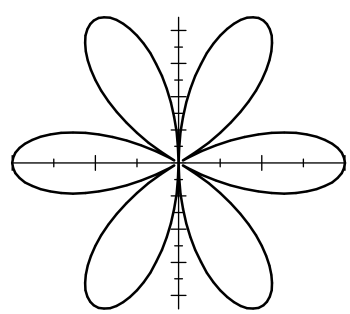
2. Which of the following is the polar plot of $r = \sin(3\theta)$?

a. 

b.  ←Correct Choice

c. 

d. 

e. 

SOLUTION: (c) is the rectangular plot of $r = \sin(3\theta)$. (b) is its polar plot because there are 3 positive loops and 3 negative loops which retrace the positive loops starting with $r = 0$ when $\theta = 0$.

3. Find the mass of a triangular plate whose vertices are $(0,0)$, $(1,0)$ and $(1,3)$, if the density is $\rho = 2y$.

a. 1

b. 2

c. 3 Correct Choice

d. 4

e. 5

SOLUTION: $M = \int\int \rho\, dA = \int_0^1 \int_0^{3x} 2y\, dy\, dx = \int_0^1 [y^2]_{y=0}^{3x}\, dx = \int_0^1 9x^2\, dx = [3x^3]_0^1 = 3$

4. Find the y-component of the center of mass of a triangular plate whose vertices are $(0,0)$, $(1,0)$ and $(1,3)$, if the density is $\rho = 2y$.

a. $\frac{9}{2}$

b. $\frac{7}{2}$

c. $\frac{5}{2}$

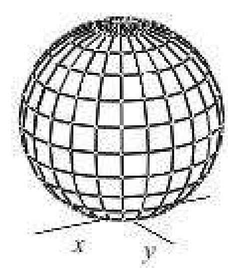
d. $\frac{3}{2}$ Correct Choice

e. $\frac{1}{2}$

SOLUTION: $M_x = \int\int y\rho\, dA = \int_0^1 \int_0^{3x} 2y^2\, dy\, dx = \int_0^1 \left[\frac{2}{3}y^3\right]_{y=0}^{3x} dx = \int_0^1 18x^3\, dx = \left[\frac{9}{2}x^4\right]_0^1 = \frac{9}{2}$

$\bar{y} = \frac{M_y}{M} = \frac{9}{2}\frac{1}{3} = \frac{3}{2}$

5. In spherical coordinates $\rho = 2\cos\varphi$ is a sphere of radius 1 centered at $(0,0,1)$. If its volume density is $\delta = x^2 + y^2 + z^2$ then its mass is given by the integral:



a. $M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{\pi} \rho^2 2\cos\varphi \sin\varphi\, d\rho\, d\varphi\, d\theta$

b. $M = \int_0^{2\pi} \int_0^{\pi} \int_0^{2\cos\varphi} \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta$

c. $M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

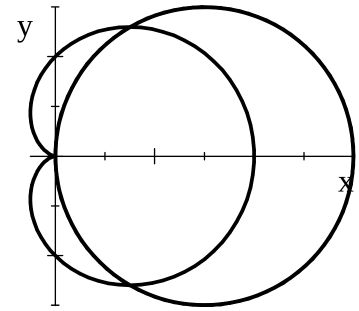
d. $M = \int_0^{2\pi} \int_0^{\pi} \int_0^{2\cos\varphi} \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

e. $M = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^4 \sin\varphi\, d\rho\, d\varphi\, d\theta$ Correct Choice

SOLUTION: $\delta = x^2 + y^2 + z^2 = \rho^2$ $\quad 0 \le \varphi \le \pi/2$ because the sphere is above the xy-plane.

$M = \iiint \delta\, dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\varphi} \rho^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$

6. Find the area inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.



a. $\frac{\pi}{4}$

b. $\frac{\pi}{2}$

c. π Correct Choice

d. $\frac{3\pi}{2}$

e. 2π

SOLUTION: Find the angles of intersection: $3\cos\theta = 1 + \cos\theta \qquad \cos\theta = \frac{1}{2} \qquad \theta = \pm\frac{\pi}{3}$

$$A = \int\int 1\,dA = 2\int_0^{\pi/3}\int_{1+\cos\theta}^{3\cos\theta} 1\,r\,dr\,d\theta = \int_0^{\pi/3}[r^2]_{1+\cos\theta}^{3\cos\theta}\,d\theta = \int_{-\pi/3}^{\pi/3} 9\cos^2\theta - (1+\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi/3} 9\cos^2\theta - (1 + 2\cos\theta + \cos^2\theta)\,d\theta = \int_0^{\pi/3} 4(1+\cos(2\theta)) - 1 - 2\cos\theta\,d\theta$$

$$= \int_0^{\pi/3} 3 + 4\cos(2\theta) - 2\cos\theta\,d\theta = [3\theta + 2\sin(2\theta) - 2\sin\theta]_0^{\pi/3} = \pi + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3}$$

$$= \pi + \sqrt{3} - \sqrt{3} = \pi$$

7. Parabolic coordinates are given by $u = y - x^2$ and $v = y + x^2$ where $v > u$. So the area element is $dA = dx\,dy =$

a. $\frac{1}{2\sqrt{2}\sqrt{v-u}}\,du\,dv$ Correct Choice

b. $\frac{-1}{2\sqrt{2}\sqrt{v-u}}\,du\,dv$

c. $\frac{1}{4\sqrt{2}\sqrt{v-u}}\,du\,dv$

d. $\frac{-1}{4\sqrt{2}\sqrt{v-u}}\,du\,dv$

e. $\frac{1}{8\sqrt{2}\sqrt{v-u}}\,du\,dv$

SOLUTION: $u + v = 2y \qquad v - u = 2x^2 \qquad x = \frac{\sqrt{v-u}}{\sqrt{2}} \qquad y = \frac{u+v}{2}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{v-u}} & \frac{1}{2} \\ \frac{1}{2\sqrt{2}}\frac{1}{\sqrt{v-u}} & \frac{1}{2} \end{vmatrix} = \frac{1}{4\sqrt{2}}\frac{-1}{\sqrt{v-u}} - \frac{1}{4\sqrt{2}}\frac{1}{\sqrt{v-u}} = \frac{1}{2\sqrt{2}}\frac{-1}{\sqrt{v-u}}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{2\sqrt{2}\sqrt{v-u}} \qquad dA = \frac{1}{2\sqrt{2}\sqrt{v-u}}\,du\,dv$$

8. If $f = xe^{yz} - ye^{xz}$, then $\vec{\nabla} \times \vec{\nabla} f =$

a. $(2xe^{yz} - 2xe^{xz} + 2xyze^{yz}, 0, 2ze^{xz} - 2ze^{yz})$
b. $(2xe^{yz} + 2xe^{xz} + 2xyze^{yz}, 0, 2ze^{xz} + 2ze^{yz})$
c. $(e^{yz} - yze^{xz}, -xze^{yz} + e^{xz}, xye^{yz} - xye^{xz})$
d. $(e^{yz} - yze^{xz}, xze^{yz} - e^{xz}, xye^{yz} - xye^{xz})$
e. $\vec{0}$ Correct Choice

SOLUTION: $\vec{\nabla} \times \vec{\nabla} f == \vec{0}$ for any twice differentiable vector field.

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (12 points) Determine whether or not each of these limits exists. If it exists, find its value.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2y^2}{x^6 + 3y^3}$

SOLUTION: Straight line approaches: $y = mx$

$$\lim_{\substack{y=mx \\ x\to 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^2}{x^6 + 3m^3x^3} = \lim_{x\to 0} \frac{3m^2x}{x^3 + 3m^3} = \frac{0}{3m^3} = 0$$

Quadratic approaches: $y = mx^2$

$$\lim_{\substack{y=mx^2 \\ x\to 0}} \frac{3x^2y^2}{x^6 + 3y^3} = \lim_{x\to 0} \frac{3x^2m^2x^4}{x^6 + 3m^3x^6} = \lim_{x\to 0} \frac{3m^2}{1 + 3m^3} \neq 0 \qquad \text{if } m \neq 0.$$

Limit does not exist because these are different.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2 + y^2}$

SOLUTION: Switch to polar: $x = r\cos\theta \quad y = r\sin\theta$

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2} = \lim_{\substack{r\to 0 \\ \theta \text{ arbitrary}}} r\cos\theta\sin^2\theta = 0$$

because $r \to 0$ while $\cos\theta\sin^2\theta$ is bounded: $-1 \le \cos\theta\sin^2\theta \le 1$.

10. (20 points) Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for the vector field $\vec{F} = (yz, -xz, z^2)$ over the paraboloid $z = 9 - x^2 - y^2$ for $z \geq 5$ oriented down and in.

Note: The paraboloid may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 9 - r^2)$.

SOLUTION: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz & -xz & z^2 \end{vmatrix} = \hat{\imath}(0 - -x) - \hat{\jmath}(0 - y) + \hat{k}(-z - z) = (x, y, -2z)$

$\left(\vec{\nabla} \times \vec{F}\right)\left(\vec{R}(r,\theta)\right) = (r\cos\theta, r\sin\theta, -2(9 - r^2)) = (r\cos\theta, r\sin\theta, 2r^2 - 18)$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\quad \cos\theta & \sin\theta & -2r) \\ \vec{e}_\theta = & (-r\sin\theta & r\cos\theta & 0) \end{array}$$

$\vec{N} = \hat{\imath}(0 - -2r^2\cos\theta) - \hat{\jmath}(0 - 2r^2\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (2r^2\cos\theta, 2r^2\sin\theta, r)$ up and out

Reverse $\vec{N} = (-2r^2\cos\theta, -2r^2\sin\theta, -r)$ now down and in

$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -2r^3\cos^2\theta - 2r^3\sin^2\theta - r(2r^2 - 18) = -4r^3 + 18r \qquad 9 - r^2 = 5 \qquad r = 2$

$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 -4r^3 + 18r\,dr\,d\theta = 2\pi[-r^4 + 9r^2]_0^2 = 2\pi(-16 + 36) = 40\pi$

11. (20 points) Compute $\iiint \vec{\nabla} \cdot \vec{F}\,dV$ for the vector field $\vec{F} = (x^3, y^3, x^2z + y^2z)$ over the solid region below the cone $z = 9 - \sqrt{x^2 + y^2}$ and above the plane $z = 5$.

SOLUTION: $\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4(x^2 + y^2) = 4r^2 \qquad 5 = 9 - r \qquad r = 4$

$\iiint \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi}\int_0^4\int_5^{9-r} 4r^2\,r\,dz\,dr\,d\theta = 2\pi\int_0^4 [4r^3 z]_{z=5}^{9-r}\,dr = 2\pi\int_0^4 4r^3(4 - r)\,dr$

$= 8\pi\left[r^4 - \frac{r^5}{5}\right]_0^4 = 8\pi 4^4\left(1 - \frac{4}{5}\right) = \frac{2048\pi}{5}$