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MATH 251 Exam 1 Fall 2005

Sections 503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Find the angle between the vectors $\vec{u} = \langle 1, -1, 1 \rangle$ and $\vec{v} = \langle 1, 2, 1 \rangle$.

a. $0°$

b. $30°$

c. $45°$

d. $60°$

e. $90°$ Correct Choice

$\vec{u} \cdot \vec{v} = 1 - 2 + 1 = 0 \qquad \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} = 0 \qquad \theta = 90°$

2. If $p(x,y) = e^{x/y}$, find $\dfrac{\partial^2 p}{\partial y^2}$.

a. $\dfrac{x}{y^2} e^{x/y}$

b. $\dfrac{-1}{y^2} e^{x/y}$

c. $\left(\dfrac{2x}{y^3} + \dfrac{x^2}{y^4} \right) e^{x/y}$ Correct Choice

d. $\left(\dfrac{2}{y^3} + \dfrac{1}{y^4} \right) e^{\frac{x}{y}}$

e. $\left(\dfrac{2}{y^3} - \dfrac{1}{y^4} \right) e^{\frac{x}{y}}$

Chain Rule and Product Rule:

$$\frac{\partial}{\partial y} e^{x/y} = -\frac{x}{y^2} e^{\frac{x}{y}}$$

$$\frac{\partial^2}{\partial y^2} e^{x/y} = \frac{2x}{y^3} e^{\frac{x}{y}} + \frac{x^2}{y^4} e^{\frac{x}{y}} = \left(\frac{2x}{y^3} + \frac{x^2}{y^4} \right) e^{\frac{x}{y}}$$

3. Find the line through the point $P = (2,-1,4)$ in the direction $\vec{v} = (1,3,-2)$.
Where does this line intersect the xy-plane?

a. $(2,-1,0)$

b. $(4,5,0)$ Correct Choice

c. $(3,2,2)$

d. $(3,2,0)$

e. $(2,2.5,0)$

The line is $X = P + t\vec{v}$, or $(x,y,z) = (2,-1,4) + t(1,3,-2) = (2+t,-1+3t,4-2t)$.
It intersects the xy-plane when $z = 4-2t = 0$ or $t = 2$.
So $x = 2+t = 4$ and $y = -1+3t = 5$.

4. At $t = 4$ the velocity of a fly is $\vec{v} = (0,2,1)$, and its acceleration is $\vec{a} = (1,0,1)$.
Find the unit binormal vector $\hat{B}$ to it's path.

a. $\left(\frac{2}{3},\frac{-1}{3},\frac{2}{3}\right)$

b. $(2,-1,-2)$

c. $\left(\frac{2}{3},\frac{-1}{3},\frac{-2}{3}\right)$

d. $(2,1,-2)$

e. $\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$ Correct Choice

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \hat{\imath}(2-0) - \hat{\jmath}(0-1) + \hat{k}(0-2) = (2,1,-2) \qquad |\vec{v}\times\vec{a}| = \sqrt{4+1+4} = 3$$

$$\hat{B} = \frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|} = \frac{1}{3}(2,1,-2) = \left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$$

5. Find the equation of the plane tangent to the graph of $z = x\cos y - \sin(xy)$ at the point where $(x,y) = (1,\pi)$. Where does this plane intersect the z-axis?

a. -1

b. 0

c. 1

d. π

e. -2π Correct Choice

$f(x,y) = x\cos y - \sin(xy) \qquad f(1,\pi) = \cos\pi - \sin(\pi) = -1$

$f_x(x,y) = \cos y - y\cos(xy) \qquad f_x(1,\pi) = \cos\pi - \pi\cos(\pi) = \pi - 1$

$f_y(x,y) = -x\sin y - x\cos(xy) \qquad f_y(1,\pi) = -\sin\pi - \cos(\pi) = 1$

$z = f(1,\pi) + f_x(1,\pi)(x-1) + f_y(1,\pi)(y-\pi) = -1 + (\pi-1)(x-1) + (y-\pi)$

$z = (\pi-1)x + y - 2\pi$ When $x = 0$ and $y = 0$, we have $z = -2\pi$.

6. Find the equation of the plane tangent to the surface $x^2z + y^2z^2 = 5$ at the point $(2,-1,1)$.

- **a.** $2x - y + 3z = 8$ Correct Choice
- **b.** $2x - y + 3z = -8$
- **c.** $2x + y + 3z = 6$
- **d.** $2x + y + 3z = -6$
- **e.** $4x - 2y + 6z = -8$

Let $f(x,y,z) = x^2z + y^2z^2$. Then $\vec{\nabla}f = (2xz, 2yz^2, x^2 + 2y^2z)$, and $\vec{N} = \vec{\nabla}f\Big|_{(2,-1,1)} = (4,-2,6)$.

The tangent plane is $\vec{N} \cdot X = \vec{N} \cdot P$ where $X = (x,y,z)$ and $\vec{P} = (2,-1,1)$.

$4x - 2y + 6z = 8 + 2 + 6 = 16$ or $2x - y + 3z = 8$

7. Duke Skywater is travelling through the galaxy. At the point with galactic coordinates $(40,25,53)$ (in lightyears), he measures the polaron density to be $U = 4300$ polarons/cm^3 and its gradient to be $\vec{\nabla}U = (3,2,1)$ polarons/cm^3/lightyear. Use this information to estimate the polaron density at the point with galactic coordinates $(42,26,52)$.

- **a.** 4291
- **b.** 4293
- **c.** 4307 Correct Choice
- **d.** 4309
- **e.** 4311

$$U(x,y,z) \approx U_{\tan}(x,y,z) = U(40,25,53) + \frac{\partial U}{\partial x}(x-40) + \frac{\partial U}{\partial y}(y-25) + \frac{\partial U}{\partial z}(z-53)$$
$$= 4300 + 3(x-40) + 2(y-25) + 1(z-53)$$

$$U(42,26,52) \approx 4300 + 3(42-40) + 2(26-25) + 1(52-53) = 4300 + 3(2) + 2(1) + 1(-1) = 4307$$

Work Out: (20 points each. Part credit possible. Show all work.)

8. (20 points) The temperature in a frying pan is given by $T(x,y) = 30 - \frac{x^2}{4} - \frac{y^4}{100}$ where distance is in cm and temperature is in $°C$. An ant is currently located at the point $(2,5)$ cm and has velocity $\vec{v} = (0.3, 0.1)$ cm/sec.

a. What is the time rate of change of the temperature as seen by the ant?

$$\vec{\nabla}T = \left(\frac{-x}{2}, \frac{-y^3}{25}\right) \qquad \vec{\nabla}T\Big|_{(2,5)} = (-1,-5)$$

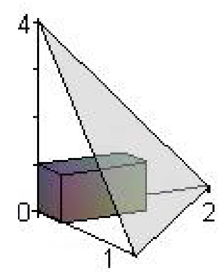
$$\frac{dT}{dt} = \vec{v} \cdot \vec{\nabla}T = (0.3, 0.1) \cdot (-1,-5) = -0.3 - 0.5 = -0.8 \frac{°C}{\text{sec}}$$

b. In what direction should the ant walk to decrease the temperature as fast as possible.

The temperature INCREASES as fast as possible in the direction $\vec{\nabla}T$.

So it DECREASES as fast as possible in the direction $-\vec{\nabla}T = (1,5)$.

9. (20 points) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $x + \frac{y}{2} + \frac{z}{4} = 1$.



$$x = 1 - \frac{y}{2} - \frac{z}{4}$$

$$V = xyz = \left(1 - \frac{y}{2} - \frac{z}{4}\right)yz = yz - \frac{1}{2}y^2z - \frac{1}{4}yz^2$$

$$V_y = z - yz - \frac{1}{4}z^2 = 0 \qquad V_z = y - \frac{1}{2}y^2 - \frac{1}{2}yz = 0$$

$$V_y = z\left(1 - y - \frac{1}{4}z\right) = 0 \qquad V_z = y\left(1 - \frac{1}{2}y - \frac{1}{2}z\right) = 0$$

If y or z is 0, then the volume is 0 and this cannot be the maximum volume.

So we solve $1 - y - \frac{1}{4}z = 0$ and $1 - \frac{1}{2}y - \frac{1}{2}z = 0$.

Rearrange the equations and double the second equation:

$$y + \frac{1}{4}z = 1 \qquad y + z = 2$$

Subtract:

$$\frac{3}{4}z = 1 \qquad z = \frac{4}{3} \qquad y = 2 - z = \frac{2}{3} \qquad x = 1 - \frac{y}{2} - \frac{z}{4} = 1 - \frac{1}{3} - \frac{1}{3} = \frac{1}{3}$$

$$V = xyz = \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{4}{3} = \frac{8}{27}$$

10. (20 points) Determine whether each of the following limits exists and say why or why not. If the limit exists, find it.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ x=r\cos\theta, y=r\sin\theta}} \frac{xy^2}{x^2+y^2} = \lim_{r\to0} \frac{r\cos\theta\, r^2\sin^2\theta}{r^2} = \lim_{r\to0} r\cos\theta\sin^2\theta = 0$$

independent of the behavior of θ. So the limit exists and equals 0.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{xy}{x^2+y^2} = \lim_{x\to0} \frac{xmx}{x^2+m^2x^2} = \frac{m}{1+m^2}$$ which is different for different m's.

So the limit does not exist.