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MATH 251 Exam 2 Fall 2005

Sections 503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Find the volume of the solid under $z = 2x^2y$ above the region in the xy-plane between $y = x$ and $y = x^2$.

a. $\frac{2}{35}$ Correct Choice
b. $\frac{35}{12}$
c. $\frac{12}{35}$
d. $\frac{1}{35}$
e. $\frac{1}{12}$

$$V = \iint 2x^2y\,dA = \int_0^1 \int_{x^2}^x 2x^2y\,dy\,dx == \int_0^1 \left[x^2y^2\right]_{y=x^2}^x dx = \int_0^1 (x^4 - x^6)\,dx = \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_{x=0}^1$$

$$= \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}$$

2. Compute $\iint \sin(x^2)\,dx\,dy$ over the triangle with vertices $(0,0)$, $(\sqrt{\pi},0)$, $(\sqrt{\pi},\sqrt{\pi})$.

a. $-\pi$
b. $-\sqrt{\pi}$
c. 1 Correct Choice
d. $\sqrt{\pi}$
e. π

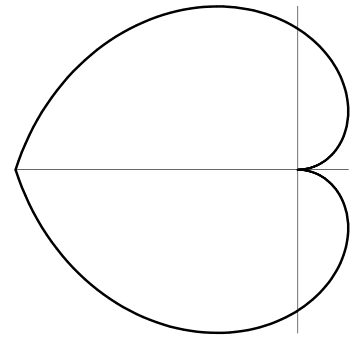
You must do the y-integral first because you don't know the antiderivative of $\sin(x^2)$.

The edges are $y = 0$, $x = \sqrt{\pi}$, $y = x$.

$$\iint \sin(x^2)\,dx\,dy = \int_0^{\sqrt{\pi}} \int_0^x \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}} \left[y\sin(x^2)\right]_{y=0}^x dx = \int_0^{\sqrt{\pi}} x\sin(x^2)\,dx$$

$$= \left[\frac{-1}{2}\cos(x^2)\right]_{x=0}^{\sqrt{\pi}} = \frac{1}{2} - -\frac{1}{2} = 1$$

3. Find the area of the heart shaped region inside the polar curve $r = |\theta|$.



a. $\frac{\pi^3}{6}$

b. $\frac{\pi^3}{3}$ Correct Choice

c. $\frac{4\pi^3}{3}$

d. $\frac{8\pi^3}{3}$

e. $\frac{16\pi^3}{3}$

Double the upper half:

$$A = 2\iint 1\,dA = 2\int_0^{\pi}\int_0^{\theta} r\,dr\,d\theta = 2\int_0^{\pi}\left[\frac{r^2}{2}\right]_{r=0}^{\theta} d\theta = 2\int_0^{\pi}\left(\frac{\theta^2}{2}\right) d\theta = 2\left[\frac{\theta^3}{6}\right]_{\theta=0}^{\pi} = \frac{\pi^3}{3}$$

4. Compute $\iiint \nabla \cdot \vec{F}\,dV$ on the solid cylinder bounded by

$x^2 + y^2 = 9,\ \ z = 0$ and $z = 5$ for the vector field $\vec{F} = (x^3, y^3, z(x^2 + y^2))$.

a. 45π
b. 90π
c. 360π
d. 810π Correct Choice
e. 900π

$$\nabla \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4x^2 + 4y^2 = 4r^2$$

$$\iiint \nabla \cdot \vec{F}\,dV = \int_0^5\int_0^{2\pi}\int_0^3 4r^2\,r\,dr\,d\theta\,dz = 5 \cdot 2\pi\left[r^4\right]_{r=0}^{3} = 810\pi$$

5. The solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ has density $\delta = z$. Find the total mass.

a. $\pi/2$
b. π
c. 2π
d. 4π Correct Choice
e. 8π

In spherical coordinates, $\delta = z = \rho\cos\varphi$ and $J = \rho^2 \sin\varphi$.

$$M = \iiint \rho \, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{\rho^4}{4}\right]_{\rho=0}^{2}\left[\frac{\sin^2\varphi}{2}\right]_{\varphi=0}^{\pi/2} = 4\pi$$

6. The solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ has density $\delta = z$.

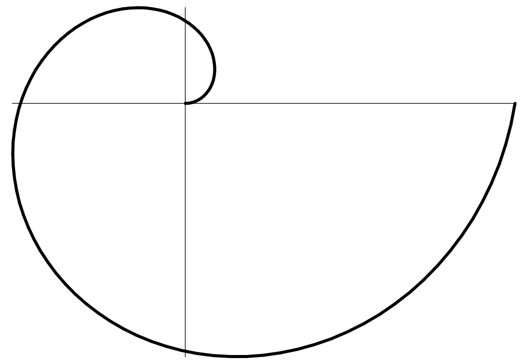
Find the z-component of the center of mass.

a. 1
b. $\frac{32}{15}\pi$
c. $\frac{64}{15}\pi$
d. $\frac{8}{15}$
e. $\frac{16}{15}$ Correct Choice

$$M_{xy} = \iiint z\rho \, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{\rho^5}{5}\right]_{\rho=0}^{2}\left[\frac{-\cos^3\varphi}{3}\right]_{\varphi=0}^{\pi/2} = \frac{64}{15}\pi$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{64\pi}{15 \cdot 4\pi} = \frac{16}{15}$$

7. Compute $\int \sqrt{1+x^2+y^2}\, ds$ along the spiral $\vec{r}(t) = (t\cos t, t\sin t)$ from $(0,0)$ to $(2\pi, 0)$.



a. $\pi + \dfrac{\pi^3}{3}$

b. $2\pi + \dfrac{8\pi^3}{3}$ Correct Choice

c. $\dfrac{(1+4\pi^2)^{3/2}}{3}$

d. $\dfrac{2(1+4\pi^2)^{3/2}}{3}$

e. $\dfrac{(1+4\pi^2)^{3/2}-1}{3}$

$\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$

$|\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{\cos^2 t + t^2\sin^2 t + \sin^2 t + t^2\cos^2 t} = \sqrt{1+t^2}$

$\sqrt{1+x^2+y^2} = \sqrt{1+t^2\cos^2 t + t^2\sin^2 t} = \sqrt{1+t^2}$

$$\int \sqrt{1+x^2+y^2}\, ds = \int_0^{2\pi} \sqrt{1+t^2}\, |\vec{v}|\, dt = \int_0^{2\pi} \sqrt{1+t^2}^{\,2} dt = \int_0^{2\pi} (1+t^2) dt$$

$$= \left[t + \frac{t^3}{3} \right]_0^{2\pi} = 2\pi + \frac{8\pi^3}{3}$$

8. Compute $\int y\,dx - x\,dy$ along the spiral $\vec{r}(t) = (t\cos t, t\sin t)$ from $(0,0)$ to $(2\pi, 0)$.

a. $\dfrac{-8\pi^3}{3}$ Correct Choice

b. $\dfrac{-4\pi^3}{3}$

c. $2\pi + \dfrac{8\pi^3}{3}$

d. $\dfrac{4\pi^3}{3}$

e. $\dfrac{8\pi^3}{3}$

$\int y\,dx - x\,dy = \int \vec{F} \cdot d\vec{s} = \int \vec{F} \cdot \vec{v}\, dt$

where $\vec{F} = (y, -x) = (t\sin t, -t\cos t)$ and $\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$.

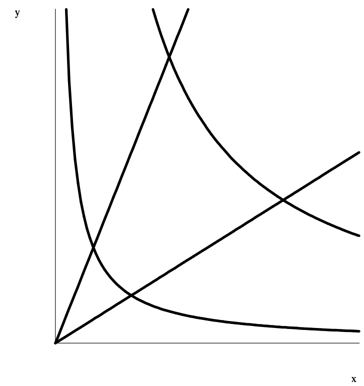
$\vec{F} \cdot \vec{v} = t\sin t(\cos t - t\sin t) - t\cos t(\sin t + t\cos t) = -t^2\sin^2 t - t^2\cos^2 t = -t^2$

$$\int y\,dx - x\,dy = -\int_0^{2\pi} t^2\, dt = \left.\frac{-t^3}{3}\right|_0^{2\pi} = \frac{-8\pi^3}{3}$$

Work Out: (Part credit possible. Show all work.)

9. (15 points) Compute $\iint_R y^2 \, dx\,dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{9}{x}, \qquad y = x, \qquad y = 4x$$



FULL CREDIT for integrating in the curvilinear coordinates (u, v) where $u^2 = xy$ and $v^2 = \frac{y}{x}$. (Solve for x and y.)

HALF CREDIT for integrating in rectangular coordinates.

$$\left\{ \begin{array}{l} u^2 = xy \\ v^2 = \frac{y}{x} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u^2 v^2 = y^2 \\ \frac{u^2}{v^2} = x^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \frac{u}{v} \\ y = uv \end{array} \right\}$$

$$J = \left| \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \right| = \left\| \begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{array} \right\| = \left| \frac{u}{v} - -\frac{u}{v} \right| = \frac{2u}{v}$$

$xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1 \qquad xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3 \qquad$ So: $1 \le u \le 3$

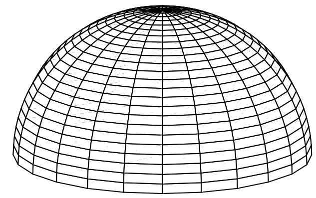
$\frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1 \qquad \frac{y}{x} = 4 \Rightarrow v^2 = 4 \Rightarrow v = 2 \qquad$ So: $1 \le v \le 2$

$$\iint_R y^2 \, dx\,dy = \int_1^2 \int_1^3 u^2 v^2 \, \frac{2u}{v} \, du \, dv = 2 \int_1^2 \int_1^3 u^3 v \, du \, dv$$

$$= 2\left[\frac{u^4}{4} \right]_{u=1}^{3} \left[\frac{v^2}{2} \right]_{v=1}^{2} = 2\left[\frac{81}{4} - \frac{1}{4} \right]\left[\frac{4}{2} - \frac{1}{2} \right] = 60$$

10. (20 points) Consider the hemispherical surface

$$z = \sqrt{4 - x^2 - y^2}$$



which may be parametrized by

$$\vec{R}(\varphi,\theta) = (2\sin\varphi\cos\theta, 2\sin\varphi\sin\theta, 2\cos\varphi).$$

Find each of the following:

a. (2 pts) $\vec{e}_\varphi = (2\cos\varphi\cos\theta, \quad 2\cos\varphi\sin\theta, \quad -2\sin\varphi)$

b. (2 pts) $\vec{e}_\theta = (-2\sin\varphi\sin\theta, \quad 2\sin\varphi\cos\theta, \quad 0)$

c. (3 pts) $\vec{N} = \hat{\imath}(4\sin^2\varphi\cos\theta) - \hat{\jmath}(-4\sin^2\varphi\sin\theta) + \hat{k}(4\sin\varphi\cos\varphi\cos^2\theta + 4\sin\varphi\cos\varphi\sin^2\theta)$

$$= (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$$

d. (2 pts) $\left|\vec{N}\right| = \sqrt{16\sin^4\varphi\cos^2\theta + 16\sin^4\varphi\sin^2\theta + 16\sin^2\varphi\cos^2\varphi}$

$$= 4\sqrt{\sin^4\varphi + \sin^2\varphi\cos^2\varphi} = 4\sin\varphi$$

e. (5 pts) The total mass of the surface if the surface density is $\delta = z$.

$$M = \int\int \delta\, dS = \int\int z\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 2\cos\varphi\, 4\sin\varphi\, d\varphi\, d\theta = 2\pi \cdot 8 \cdot \frac{\sin^2\varphi}{2}\Bigg|_{\varphi=0}^{\pi/2} = 8\pi$$

f. (6 pts) The z-component of the center of mass of the surface if the surface density is $\delta = z$.

$$M_{xy} = \iint z\delta\, dS = \iint z^2\left|\vec{N}\right| d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} 4\cos^2\varphi\, 4\sin\varphi\, d\varphi\, d\theta$$

$$= 2\pi \cdot 16 \cdot \frac{-\cos^3\varphi}{3}\Bigg|_{\varphi=0}^{\pi/2} = \frac{32\pi}{3}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3 \cdot 8\pi} = \frac{4}{3}$$

11. (10 points) Consider the vector field $\vec{F} = (-y^3, x^3, z(x^2+y^2))$ on the hemispherical surface of problem 10. Find each of the following:

a. (3 pts) $\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y^3 & x^3 & (x^2+y^2)z \end{vmatrix}$

$$= \hat{\imath}(2yz) - \hat{\jmath}(2xz) + \hat{k}(3x^2+3y^2) = (2yz, -2xz, 3x^2+3y^2)$$

b. (2 pts) $\nabla \times \vec{F}\left(\vec{R}(\varphi,\theta)\right) =$

$$= (8\sin\varphi\cos\varphi\sin\theta, -8\sin\varphi\cos\varphi\cos\theta, 3\cdot 4\sin^2\varphi\cos^2\theta + 3\cdot 4\sin^2\varphi\sin^2\theta)$$

$$= (8\sin\varphi\cos\varphi\sin\theta, -8\sin\varphi\cos\varphi\cos\theta, 12\sin^2\varphi)$$

c. (5 pts) $\iint \nabla \times \vec{F} \cdot d\vec{S}$ with normal pointing up.

From problem 10, $\vec{N} = (4\sin^2\varphi\cos\theta, 4\sin^2\varphi\sin\theta, 4\sin\varphi\cos\varphi)$ which points up.

$$\iint \nabla \times \vec{F} \cdot d\vec{S} = \iint \nabla \times \vec{F} \cdot \vec{N}\, d\varphi\, d\theta$$

$$= \iint (32\sin^3\varphi\cos\varphi\sin\theta\cos\theta - 32\sin^3\varphi\cos\varphi\sin\theta\cos\theta + 48\sin^3\varphi\cos\varphi)\, d\varphi\, d\theta$$

$$= 48\int_0^{2\pi}\int_0^{\pi/2} \sin^3\varphi\cos\varphi\, d\varphi\, d\theta = 48\cdot 2\pi\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2} = 24\pi$$

12. (10 points) A bowl has the shape $z = \dfrac{x^2+y^2}{3}$ for $z \le 3$.

The bowl is filled with liquid of density $\rho = 12 - 2z$.

Find the total mass of the liquid.

$$M = \iiint \rho\, dV = \int_0^{2\pi}\int_0^3\int_{r^2/3}^3 (12-2z)\, r\, dz\, dr\, d\theta = 2\pi\int_0^3 \left[12z - z^2\right]_{z=r^2/3}^3 r\, dr$$

$$= 2\pi\int_0^3 \left[(36-9) - \left(4r^2 - \frac{r^4}{9}\right)\right] r\, dr = 2\pi\int_0^3 \left[27r - 4r^3 + \frac{r^5}{9}\right] dr$$

$$= 2\pi\left[\frac{27r^2}{2} - r^4 + \frac{r^6}{54}\right]_{r=0}^3 = 2\pi\left(\frac{243}{2} - 81 + \frac{27}{2}\right) = 108\pi$$