Name____________________ ID______________

1	/ 5
8	/20
9	/20
10	/20
Total	/25

MATH 251 Quiz 4 Fall 2005

Sections 503 Solutions P. Yasskin

You must do Problem 1.
Then do one (only) of 8, 9, or 10.

1. (5 points) Find the mass of the solid below $z = x^2 y$ above the region in the xy-plane between $y = x$ and $y = x^2$ if the density is $\rho(x,y,z) = 6z$.

$$M = \iiint \rho\, dV = \int_0^1 \int_{x^2}^{x} \int_0^{x^2 y} 6z\, dz\, dy\, dx = \int_0^1 \int_{x^2}^{x} \left[3z^2\right]_{z=0}^{x^2 y} dy\, dx = \int_0^1 \int_{x^2}^{x} 3x^4 y^2\, dy\, dx$$

$$= \int_0^1 \left[x^4 y^3\right]_{y=x^2}^{x} dx = \int_0^1 (x^7 - x^{10})\, dx = \left[\frac{x^8}{8} - \frac{x^{11}}{11}\right]_{x=0}^{1} = \frac{1}{8} - \frac{1}{11} = \frac{11-8}{88} = \frac{3}{88}$$

8. (20 points) The carbon monoxide density in the air on a certain highway is given by $\rho = \dfrac{6xy^2}{z}$ where distances are measured in feet and density is measured in parts per million.

a. If a bird is at the point $(4,3,2)$ and its velocity is $(-3,4,1)$, does the bird feel the CO density increasing or decreasing and how fast?

$$\vec{\nabla}\rho = \left(\frac{6y^2}{z}, \frac{12xy}{z}, \frac{-6xy^2}{z^2}\right) \qquad \vec{\nabla}\rho\Big|_{(4,3,2)} = (27,72,-54)$$

$$\frac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = (-3,4,1) \cdot (27,72,-54) = -81 + 288 - 54 = 153 \qquad \text{increasing}$$

b. Use the linear approximation to estimate the CO density at the point $(3.97, 3.04, 2.01)$.

$$\rho(x,y,z) \approx \rho(a,b,c) + \rho_x|_{(a,b,c)}(x-a) + \rho_y|_{(a,b,c)}(y-b) + \rho_z|_{(a,b,c)}(z-c)$$

$$(a,b,c) = (4,3,2) \qquad (x,y,z) = (3.97,3.04,2.01) \qquad (x-a, y-b, z-c) = (-.03,.04,.01)$$

$$\rho(a,b,c) = \rho(4,3,2) = \frac{6 \cdot 4 \cdot 3^2}{2} = 108 \qquad (\rho_x,\rho_y,\rho_z)|_{(a,b,c)} = \vec{\nabla}\rho\Big|_{(4,3,2)} = (27,72,-54)$$

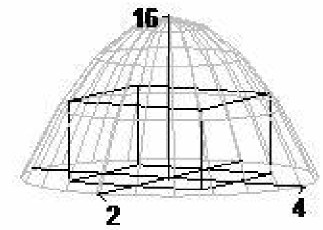
$$\rho(3.97,3.04,2.01) \approx 108 + 27(-.03) + 72(.04) - 54(.01) = 108 + 1.53 = 109.53$$

c. If a bird is at the point $(4,3,2)$, in what direction should it fly to DECREASE the CO density as fast as possible?

$$-\vec{\nabla}\rho\Big|_{(4,3,2)} = (-27,-72,54)$$

9. (20 points) Find the volume of the largest rectangular box whose base is in the xy-plane, whose sides are parallel to the coordinate planes and whose top 4 vertices are on the elliptic paraboloid

$$z = 16 - 4x^2 - y^2.$$



Take x and y in the first quadrant. Then

$V = 4xyz = 4xy(16 - 4x^2 - y^2) = 64xy - 16x^3y - 4xy^3$

$V_x = 64y - 48x^2y - 4y^3 = 0 \qquad V_y = 64x - 16x^3 - 12xy^2 = 0$

$V_x = 4y(16 - 12x^2 - y^2) = 0 \qquad V_y = 4x(16 - 4x^2 - 3y^2) = 0$

If x or y is 0, then the volume is 0 and this cannot be the maximum volume.

So we solve $16 - 12x^2 - y^2 = 0$ and $16 - 4x^2 - 3y^2 = 0$.

Multiply the first equation by 3 and subtract the second equation:

$48 - 36x^2 - 3y^2 = 0$ minus $16 - 4x^2 - 3y^2 = 0$ equals $32 - 32x^2 = 0$

So: $x = 1 \qquad y^2 = 16 - 12x^2 = 4 \qquad y = 2$

$z = 16 - 4x^2 - y^2 = 16 - 4 - 4 = 8$

$V = 4xyz = 4 \cdot 1 \cdot 2 \cdot 8 = 64$

10. (20 points) Determine whether each of the following limits exists and say why or why not. If the limit exists, find it.

a. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = \frac{0}{m^2} = 0$$ for all m's.

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2mx^2}{x^4+m^2x^4} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$ which is different for different m's.

So the limit does not exist.

b. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2}{x^4+y^2}$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=mx}} \frac{y^2}{x^4+y^2} = \lim_{x\to 0} \frac{m^2x^2}{x^4+m^2x^2} = \lim_{x\to 0} \frac{m^2}{x^2+m^2}$$ which is different for different m's.

So the limit does not exist.