1-9	/45
10	/15
11	/10
12	/20
13	/15
Total	/105

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MATH 251 Exam 2 Spring 2006

Sections 506 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\iint xy^2\,dx\,dy$ over the region bounded by $x = y^2$ and $x = 4$.

a. $\dfrac{256}{21}$

b. $\dfrac{512}{21}$ Correct Choice

c. $\dfrac{160}{21}$

d. $\dfrac{272}{15}$

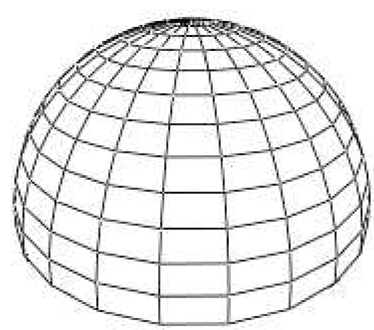
e. $\dfrac{544}{15}$

outer y-integral $-2 \le y \le 2$ $y^2 \le x \le 4$

$$\iint xy^2\,dx\,dy = \int_{-2}^{2}\int_{y^2}^{4} xy^2\,dx\,dy = \int_{-2}^{2}\left[\frac{x^2y^2}{2}\right]_{x=y^2}^{4} dy = \int_{-2}^{2}\left(8y^2 - \frac{y^6}{2}\right)dy$$

$$= \left[\frac{8y^3}{3} - \frac{y^7}{14}\right]_{-2}^{2} = 2\left(\frac{64}{3} - \frac{64}{7}\right) = 128\left(\frac{7-3}{21}\right) = \frac{512}{21}$$

2. If $\vec{F} = (xz^2, yz^2, z^3)$, compute $\iiint \vec{\nabla}\cdot F\,dV$ over the solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$.



a. $\dfrac{64\pi}{3}$ Correct Choice

b. $\dfrac{64\pi}{15}$

c. $\dfrac{32\pi}{15}$

d. $\dfrac{320\pi}{3}$

e. $\dfrac{640\pi}{3}$

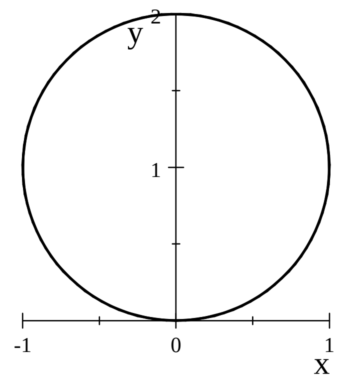
$$\vec{\nabla}\cdot\vec{F} = \partial_x(xz^2) + \partial_y(yz^2) + \partial_z(z^3) = z^2 + z^2 + 3z^2 = 5z^2$$

In spherical coordinates: $dV = \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$ $\vec{\nabla}\cdot\vec{F} = 5\rho^2\cos^2\varphi$

$$\iiint \vec{\nabla}\cdot F\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 5\rho^2\cos^2\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/2}\cos^2\varphi\sin\varphi\,d\varphi\int_0^2 5\rho^4\,d\rho$$

$$= 2\pi\left[\frac{-\cos^3\varphi}{3}\right]_0^{\pi/2}\left[\rho^5\right]_0^2 = 2\pi\left(\frac{1}{3}\right)32 = \frac{64\pi}{3}$$

3. Compute $\iint \frac{1}{y}\,dA$ over the region inside the circle given in polar coordinates by $r = 2\sin\theta$.



a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$
c. π
d. 2π Correct Choice
e. undefined

$$\iint \frac{1}{y}\,dA = \int_0^{\pi}\int_0^{2\sin\theta} \frac{1}{r\sin\theta}\,r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r}{\sin\theta}\right]_{r=0}^{2\sin\theta} d\theta = \int_0^{\pi} 2\,d\theta = \Big[2\theta\Big]_{\theta=0}^{\pi} = 2\pi$$

4. Find the mass of the solid inside the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3 - \sqrt{x^2+y^2}$ if the density is $\delta = \sqrt{x^2+y^2}$.

a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$
c. π
d. 2π
e. 8π Correct Choice

In cylindrical coordinates: $dV = r\,dr\,d\theta\,dz$ $\quad \delta = r$ $\quad z = 3 - r$

$$M = \iiint \delta\,dV = \int_0^{2\pi}\int_0^{2}\int_0^{3-r} r\,r\,dz\,dr\,d\theta = 2\pi\int_0^2 [r^2 z]_{z=0}^{3-r}\,dr = 2\pi\int_0^2 r^2(3-r)\,dr$$

$$= 2\pi\left[r^3 - \frac{r^4}{4}\right]_0^2 = 2\pi(8-4) = 8\pi$$

5. Find the z-component of the center of mass of the solid inside the cylinder $x^2 + y^2 = 4$ between $z = 0$ and $z = 3 - \sqrt{x^2+y^2}$ if the density is $\delta = \sqrt{x^2+y^2}$.

a. $\frac{32\pi}{5}$
b. $\frac{16\pi}{5}$
c. $\frac{4}{5}$ Correct Choice
d. $\frac{5}{32\pi}$
e. $\frac{5}{4}$

$$M_{xy} = \iiint z\delta\,dV = \int_0^{2\pi}\int_0^{2}\int_0^{3-r} z\,r\,r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r^2\frac{z^2}{2}\right]_{z=0}^{3-r} dr = \pi\int_0^2 r^2(3-r)^2\,dr$$

$$= \pi\int_0^2 r^2(9 - 6r + r^2)\,dr = \pi\left[3r^3 - \frac{6r^4}{4} + \frac{r^5}{5}\right]_0^2 = \frac{32\pi}{5}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{5\cdot 8\pi} = \frac{4}{5}$$

6. Compute $\int_0^2 \int_y^2 e^{x^2}\, dx\, dy$

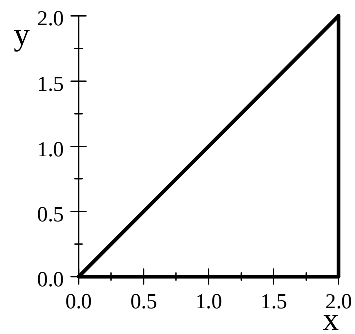
a. $\frac{1}{2}(1 - e^{-2})$
b. $\frac{1}{2}e^{-2}$
c. $\frac{1}{2}(e^2 - 1)$
d. $\frac{1}{2}e^4$
e. $\frac{1}{2}(e^4 - 1)$ Correct Choice

You don't know $\int e^{x^2}\, dx$

Reverse the order of integration:

Plot the region $0 \le y \le 2 \quad y \le x \le 2.$

Observe it is also $0 \le x \le 2 \quad 0 \le y \le x.$



$$\int_0^2 \int_y^2 e^{x^2}\, dx\, dy = \int_0^2 \int_0^x e^{x^2}\, dy\, dx = \int_0^2 \left[y e^{x^2} \right]_{y=0}^{x} dx = \int_0^2 x e^{x^2}\, dx = \left[\frac{1}{2} e^{x^2} \right]_{x=0}^{2} = \frac{1}{2}(e^4 - 1)$$

7. Compute $\oint \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (-y, x)$ along the curve $\vec{r}(t) = (\sin t \cos t, \sin^2 t)$ for $0 \le t \le \pi$. HINT: Factor $\vec{F} \cdot \vec{v}$.

a. $\frac{\pi}{4}$
b. $\frac{\pi}{2}$ Correct Choice
c. π
d. 2π
e. 4π

$\vec{v} = (\cos^2 t - \sin^2 t, 2 \sin t \cos t) \qquad \vec{F}(\vec{r}(t)) = (-\sin^2 t, \sin t \cos t)$

$\vec{F} \cdot \vec{v} = -\sin^2 t(\cos^2 t - \sin^2 t) + \sin t \cos t \cdot (2 \sin t \cos t) = -\sin^2 t \cos^2 t + \sin^4 t + 2 \sin^2 t \cos^2 t$

$\qquad = \sin^4 t + \sin^2 t \cos^2 t = \sin^2 t$

$$\oint \vec{F} \cdot d\vec{s} = \int_0^\pi \vec{F} \cdot \vec{v}\, dt = \int_0^\pi \sin^2 t\, dt = \int_0^\pi \frac{1 - \cos 2t}{2}\, dt = \left[\frac{t}{2} - \frac{\sin 2t}{4} \right]_0^\pi = \frac{\pi}{2}$$

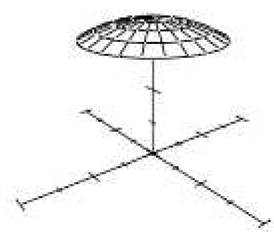
8. If $f = \sin(xyz)$ compute $\vec{\nabla} \cdot \vec{\nabla} f$.

a. 0
b. $\sin(xyz)(-y^2z^2, x^2z^2, -x^2y^2)$
c. $\sin(xyz)(-y^2z^2, -x^2z^2, -x^2y^2)$
d. $-(y^2z^2 + x^2z^2 + x^2y^2)\sin(xyz)$ Correct Choice
e. $-(y^2z^2 - x^2z^2 + x^2y^2)\sin(xyz)$

$$\vec{\nabla} f = \left(\partial_x f,\ \partial_y f,\ \partial_z f\right) = (yz\cos(xyz), xz\cos(xyz), xy\cos(xyz))$$

$$\vec{\nabla} \cdot \vec{\nabla} f = \partial_x(yz\cos(xyz)) + \partial_y(xz\cos(xyz)) + \partial_z(xy\cos(xyz))$$
$$= -y^2z^2\sin(xyz) - x^2z^2\sin(xyz) - x^2y^2\sin(xyz)$$

9. Find the area of the polar ice cap given in spherical coordinates by $0 \le \varphi \le \frac{\pi}{6}$ given that the radius of the earth is 4000 miles.
HINT: Parametrize the piece of the sphere and find the normal vector.



a. $4000^2\pi\left(2 - \sqrt{3}\right)$ Correct Choice
b. $4000^2\left(1 - \frac{\sqrt{3}}{2}\right)$
c. $4000^2\left(2 - \sqrt{3}\right)$
d. $4000^2\pi\left(1 - \frac{\sqrt{3}}{2}\right)$
e. $4000^2\pi$

$$\vec{R}(\varphi,\theta) = \left(4000\sin\varphi\cos\theta,\ 4000\sin\varphi\sin\theta,\ 4000\cos\varphi\right)$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\varphi = & 4000\cos\varphi\cos\theta & 4000\cos\varphi\sin\theta & -4000\sin\varphi \\ \vec{e}_\theta = & -4000\sin\varphi\sin\theta & 4000\sin\varphi\cos\theta & 0 \end{array}$$

$$\vec{N} = \hat{\imath}(4000^2\sin^2\varphi\cos\theta) - \hat{\jmath}(-4000^2\sin^2\varphi\sin\theta) + \hat{k}(4000^2\sin\varphi\cos\varphi\cos^2\theta + 4000^2\sin\varphi\cos\varphi\sin^2\theta)$$
$$= \left(4000^2\sin^2\varphi\cos\theta,\ 4000^2\sin^2\varphi\sin\theta,\ 4000^2\sin\varphi\cos\varphi\right)$$

$$\left|\vec{N}\right| = 4000^2\sqrt{(\sin^2\varphi\cos\theta)^2 + (\sin^2\varphi\sin\theta)^2 + (\sin\varphi\cos\varphi)^2} = 4000^2\sin\varphi$$

$$A = \iint dS = \iint\left|\vec{N}\right| d\varphi\, d\theta = 4000^2\int_0^{2\pi}\int_0^{\pi/6}\sin\varphi\, d\varphi\, d\theta = 4000^2(2\pi)[-\cos\varphi]_0^{\pi/6} = 4000^2\pi\left(2 - \sqrt{3}\right)$$

Work Out: (Part credit possible. Show all work.)

10. $\left(15\right.$ points$\left.\right)$ The average of a function f on a curve C is $f_{\text{ave}} = \frac{1}{L}\int_C f\,ds$ where L is the length of the curve. Find the average temperature T_{ave} on the curve $\vec{r}(t) = \left(t, t^2, \frac{2}{3}t^3\right)$ between $(0,0,0)$ and $\left(1, 1, \frac{2}{3}\right)$ if the temperature is $T = xy^2 + 3yz$.

The velocity is $\vec{v} = (1, 2t, 2t^2)$ $\quad |\vec{v}| = \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1 + 2t^2)^2} = 1 + 2t^2$

The length is $L = \int_C 1\,ds = \int_0^1 |\vec{v}|\,dt = \int_0^1 (1 + 2t^2)\,dt = \left[t + \frac{2}{3}t^3\right]_0^1 = 1 + \frac{2}{3} = \frac{5}{3}$

On the curve the temperature is $T = xy^2 + 3yz = (t)(t^2)^2 + 3(t^2)\left(\frac{2}{3}t^3\right) = 3t^5$

$$\int_C T\,ds = \int_0^1 T|\vec{v}|\,dt = \int_0^1 3t^5(1 + 2t^2)\,dt = \left[\frac{3t^6}{6} + \frac{6t^8}{8}\right]_0^1 = \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

So

$$T_{\text{ave}} = \frac{1}{L}\int_C T\,ds = \frac{3}{5} \cdot \frac{5}{4} = \frac{3}{4}$$

11. $\left(10\right.$ points$\left.\right)$ Compute $\iiint x\,dV$ over the triangular pyramid with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$, and $(0,0,4)$.

The boundry planes are $x = 0, y = 0, z = 0$, and $z = 4 - 4x - 2y$

When $z = 0$ the edge is $4 - 4x - 2y = 0$ or $y = 2 - 2x$

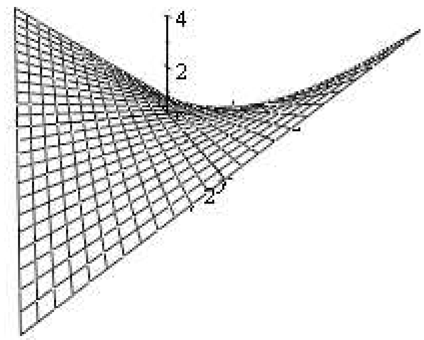
$$\iiint x\,dV = \int_0^1\int_0^{2-2x}\int_0^{4-4x-2y} x\,dz\,dy\,dx = \int_0^1\int_0^{2-2x}\Big[xz\Big]_{z=0}^{4-4x-2y} dy\,dx = \int_0^1\int_0^{2-2x} x(4 - 4x - 2y)\,dy\,dx$$

$$= \int_0^1\int_0^{2-2x}(4x - 4x^2 - 2xy)\,dy\,dx = \int_0^1 [4xy - 4x^2y - xy^2]_{y=0}^{2-2x}\,dx$$

$$= \int_0^1\left(4x(2 - 2x) - 4x^2(2 - 2x) - x(2 - 2x)^2\right)dx$$

$$= \int_0^1 (4x - 8x^2 + 4x^3)\,dx = \left[2x^2 - \frac{8}{3}x^3 + x^4\right]_0^1 = 2 - \frac{8}{3} + 1 = \frac{1}{3}$$

12. $\left(20\text{ points}\right)$ Consider the saddle $z = xy$ which may be parametrized by $\vec{R}(u,v) = (u,v,uv)$. Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-yz, xz, 0)$ over the piece of the saddle with $0 \le x \le 2$ and $0 \le y \le 3$ with normal pointing up.



HINT: First compute each of the following:

$\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \times \vec{F}\left(\vec{R}(u,v)\right)$, $\vec{e}_u$, $\vec{e}_v$, $\vec{N}$, $\vec{\nabla} \times \vec{F} \cdot \vec{N}$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & 0 \end{vmatrix} = \hat{\imath}(-x) - \hat{\jmath}(y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(u,v)\right) = (-u, -v, 2uv)$$

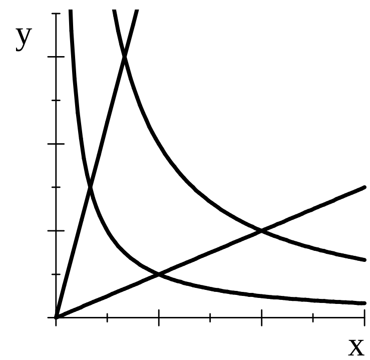
$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & 1 & 0 & v \\ \vec{e}_v = & 0 & 1 & u \end{array} \qquad \vec{N} = \vec{e}_u \times \vec{e}_v = \hat{\imath}(-v) - \hat{\jmath}(u) + \hat{k}(1) = (-v, -u, 1)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = uv + vu + 2uv = 4uv$$

$$\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^3 \int_0^2 4uv\,du\,dv = \int_0^2 2u\,du \int_0^3 2v\,dv = [u^2]_0^2 [v^2]_0^3 = 4 \cdot 9 = 36$$

13. (15 points) Compute $\iint_R \frac{y}{x}\, dx\,dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{4}{x}, \qquad y = x, \qquad y = 9x$$



FULL CREDIT for integrating in the curvilinear coordinates (u, v) where $u^2 = xy$ and $v^2 = \frac{y}{x}$. (Solve for x and y.)

HALF CREDIT for integrating in rectangular coordinates.

$$\left\{ \begin{array}{l} u^2 = xy \\ v^2 = \frac{y}{x} \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} u^2 v^2 = y^2 \\ \frac{u^2}{v^2} = x^2 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} x = \frac{u}{v} \\ y = uv \end{array} \right\}$$

$$J = \left|\left| \frac{\partial(x,y)}{\partial(u,v)} \right|\right| = \left|\left| \begin{array}{cc} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{array} \right|\right| = \left| \frac{u}{v} - -\frac{u}{v} \right| = \frac{2u}{v}$$

$xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1 \qquad xy = 4 \Rightarrow u^2 = 4 \Rightarrow u = 2 \qquad$ So: $1 \le u \le 2$

$\frac{y}{x} = 1 \Rightarrow v^2 = 1 \Rightarrow v = 1 \qquad \frac{y}{x} = 9 \Rightarrow v^2 = 9 \Rightarrow v = 3 \qquad$ So: $1 \le v \le 3$

$$\iint_R \frac{y}{x}\, dx\,dy = \int_1^3 \int_1^2 v^2 \, \frac{2u}{v} \, du \, dv = 2 \int_1^3 \int_1^2 uv \, du \, dv$$

$$= 2\left[\frac{u^2}{2} \right]_{u=1}^{2} \left[\frac{v^2}{2} \right]_{v=1}^{3} = 2\left[\frac{4}{2} - \frac{1}{2} \right]\left[\frac{9}{2} - \frac{1}{2} \right] = 12$$