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1-3	/12
4	/ 8
5	/ 8
Total	/28

MATH 251 Quiz 7 Spring 2006

Sections 506 Solutions P. Yasskin

Multiple Choice: (4 points each)

1. (4 points) If $\vec{F} = (2yz, -2xz, x^2z + y^2z)$, compute $\vec{\nabla} \cdot \vec{F}$.

a. $2yz - 2xz + 2x + 2y - 4z$

b. $x^2 + y^2$ Correct Choice

c. $(2yz + 2x, 2xz - 2y, -4z)$

d. $(0, 0, x^2 + y^2)$

e. $(2yz + 2x, 2y - 2xz, -4z)$

$$\vec{\nabla} \cdot \vec{F} = \partial_x(2yz) + \partial_y(-2xz) + \partial_z(x^2z + y^2z) = x^2 + y^2$$

2. (4 points) If $\vec{F} = (2yz, -2xz, x^2z + y^2z)$, compute $\vec{\nabla} \times \vec{F}$.

a. $2yz - 2xz + 2x + 2y - 4z$

b. $x^2 + y^2$

c. $(2yz + 2x, 2xz - 2y, -4z)$

d. $(0, 0, x^2 + y^2)$

e. $(2yz + 2x, 2y - 2xz, -4z)$ Correct Choice

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2yz & -2xz & x^2z + y^2z \end{vmatrix} = \hat{\imath}(2yz + 2x) - \hat{\jmath}(2xz - 2y) + \hat{k}(-2z - 2z)$$
$$= (2yz + 2x, 2y - 2xz, -4z)$$

3. (4 points) If $\vec{F} = (2yz, -2xz, x^2z + y^2z)$, compute $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F}$.

a. $2x - 2y$

b. $2x + 2y$

c. $(2, 2, -4)$

d. 0 Correct Choice

e. undefined

$\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any $\vec{F}$.

In particular, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = \partial_x(2yz + 2x) + \partial_y(2y - 2xz) + \partial_z(-4z) = 0$

4. (8 points) Find the mass and center of mass of a wire in the shape of the semicircle $x^2 + y^2 = 4$ with $y \ge 0$ if the density is $\rho(x,y) = y$.

Note: By symmetry $\bar{x} = 0$. So you just need to compute M and $\bar{y}$.

$$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta) \qquad \vec{v} = (-2\sin\theta, 2\cos\theta) \qquad |\vec{v}| = \sqrt{4\sin^2\theta + 4\cos^2\theta} = 2 \qquad \rho = y = 2\sin\theta$$

$$M = \int \rho\, ds = \int y|\vec{v}|\, d\theta = \int_0^\pi 2\sin\theta\, 2\, d\theta = \Big[-4\cos\theta\Big]_0^\pi = 4 - -4 = 8$$

$$M_x = \int y\rho\, ds = \int y^2 |\vec{v}|\, d\theta = \int_0^\pi 4\sin^2\theta\, 2\, d\theta = 8\int_0^\pi \frac{1-\cos 2\theta}{2}\, d\theta = 4\Big[\theta - \frac{\sin 2\theta}{2}\Big]_0^\pi = 4\pi$$

$$\bar{y} = \frac{M_x}{M} = \frac{4\pi}{8} = \frac{\pi}{2}$$

5. (8 points) Compute $\iint \vec{\nabla} \times \vec{F}\, d\vec{S}$ over the cone $z = \sqrt{x^2 + y^2}$ for $z \le 4$ with normal pointing down and out, for the vector field $\vec{F} = (2yz, -2xz, x^2z + y^2z)$.

Note: The cone may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$. Follow these steps:

$$\vec{e}_r = \big(\cos\theta,\ \sin\theta,\ 1\big)$$

$$\vec{e}_\theta = \big(-r\sin\theta,\ r\cos\theta,\ 0\big)$$

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta)$$

$$= (-r\cos\theta, -r\sin\theta, r)$$ which points up and in.

Rev: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 2yz & -2xz & x^2z + y^2z \end{vmatrix} = \hat{\imath}(2yz + 2x) - \hat{\jmath}(2xz - 2y) + \hat{k}(-2z - 2z)$$

$$= (2yz + 2x, 2y - 2xz, -4z)$$

$$\vec{\nabla} \times \vec{F}\big(\vec{R}(r,\theta)\big) = (2r^2\sin\theta + 2r\cos\theta, 2r\sin\theta - 2r^2\cos\theta, -4r)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = (2r^2\sin\theta + 2r\cos\theta)(r\cos\theta) + (2r\sin\theta - 2r^2\cos\theta)(r\sin\theta) + (-4r)(-r)$$

$$= 2r^2\cos^2\theta + 2r^2\cos^2\theta + 4r^2 = 6r^2$$

$$\iint \vec{\nabla} \times \vec{F}\, d\vec{S} = \int_0^{2\pi}\int_0^4 6r^2\, dr\, d\theta = \int_0^{2\pi} \Big[2r^3\Big]_{r=0}^4 d\theta = \int_0^{2\pi} 128\, d\theta = 256\pi$$