1. The vertices of a triangle are \( P = (3, 4, -5), \; Q = (3, 5, -4) \) and \( R = (5, 2, -5). \) Find the angle at \( P. \)

a. \( 90^\circ \)

b. \( 120^\circ \) **Correct Choice**

c. \( 135^\circ \)

d. \( 150^\circ \)

e. \( 180^\circ \)

\[
\overrightarrow{PQ} = Q - P = (0, 1, 1) \quad \overrightarrow{PR} = R - P = (2, -2, 0) \quad |\overrightarrow{PQ}| = \sqrt{2} \quad |\overrightarrow{PR}| = \sqrt{8} \quad \overrightarrow{PQ} \cdot \overrightarrow{PR} = -2
\]

\[
\cos \theta = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{|\overrightarrow{PQ}| |\overrightarrow{PR}|} = \frac{-2}{\sqrt{2} \sqrt{8}} = -\frac{1}{2} \quad \theta = 120^\circ
\]

2. Find the volume of the parallelepiped with edge vectors:

\( \vec{a} = \langle 4, 1, 2 \rangle \quad \vec{b} = \langle 2, 2, 1 \rangle \quad \vec{c} = \langle 1, 3, 0 \rangle \)

a. \(-3\)

b. \(0\)

c. \(\sqrt{3}\)

d. \(3 \) **Correct Choice**

e. \(9\)

\[
V = |\vec{a} \cdot \vec{b} \times \vec{c}| = \begin{vmatrix} 4 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = |0 + 12 - 4 - 12 - 0| = |-3| = 3
\]
3. Consider the set of all points $P$ whose distance from $(1, 0, 0)$ is 3 times its distance from $(-1, 0, 0)$. This set is a  
   a. sphere.  Correct Choice
   b. ellipsoid.
   c. hyperboloid.
   d. elliptic paraboloid.
   e. hyperbolic paraboloid.

\[
\sqrt{(x-1)^2 + y^2 + z^2} = 3\sqrt{(x+1)^2 + y^2 + z^2}
\]
\[
(x-1)^2 + y^2 + z^2 = 9(x+1)^2 + 9y^2 + 9z^2
\]
\[
0 = 8x^2 + 20x + 8y^2 + 8z^2 + 8
\]
\[
0 = x^2 + \frac{5}{2}x + y^2 + z^2 + 1 = \left(x + \frac{5}{4}\right)^2 + y^2 + z^2 - \frac{9}{16}
\]

sphere

4. For the curve $\vec{r}(t) = (\sin^2 t, \cos^2 t, \sin^2 t - \cos^2 t)$ which of the following is FALSE?  
   a. $\vec{v} = \langle 2 \sin t \cos t, -2 \sin t \cos t, 4 \sin t \cos t \rangle$
   b. $|\vec{v}| = \sqrt{24} \sin t \cos t$
   c. $\hat{T} = \left\langle \frac{2}{\sqrt{24}}, \frac{-2}{\sqrt{24}}, \frac{4}{\sqrt{24}} \right\rangle$
   d. $a_T = 0$  Correct Choice
   e. $a_N = 0$

$\vec{v}$, $|\vec{v}|$, and $\hat{T}$ are correct by computation.
Since $\hat{T}$ is constant, its direction does not change and $a_N = 0$.
Since $|\vec{v}|$ is not constant, $a_T = \frac{d|\vec{v}|}{dt} \neq 0$.

5. For the curve $\vec{r}(t) = (\sin^2 t, \cos^2 t, \sin^2 t - \cos^2 t)$ compute the arc length between $\vec{r}(0) = (0, 1, -1)$ and $\vec{r}\left(\frac{\pi}{2}\right) = (1, 0, 1)$.
   a. $\frac{1}{4} \sqrt{6}$
   b. $\frac{1}{2} \sqrt{6}$
   c. $\sqrt{6}$  Correct Choice
   d. $2\sqrt{6}$
   e. $4\sqrt{6}$

\[
L = \int_0^{\pi/2} \sqrt{24 \sin t \cos t} \, dt = \left. \sqrt{24} \frac{\sin^2 t}{2} \right|_0^{\pi/2} = \frac{1}{2} \sqrt{24} = \sqrt{6}
\]
6. The plot at the right represents which vector field?
   a. \( \vec{A} = \langle x, y \rangle \)
   b. \( \vec{B} = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right) \)
   c. \( \vec{C} = \langle y, x \rangle \)
   d. \( \vec{D} = \left( \frac{y}{\sqrt{x^2 + y^2}}, \frac{x}{\sqrt{x^2 + y^2}} \right) \) Correct Choice
   e. \( \vec{E} = \langle x + y, x - y \rangle \)

   The vectors all have the same length. So it must be one of the unit vector fields: \( \vec{B} \) or \( \vec{D} \).

   \( \vec{B} \) points radial. \( \vec{D} \) is vertical on the x-axis and horizontal on the y-axis.

7. Describe the level surfaces of \( f(x, y, z) = x^2 - y^2 - z^2 \).
   a. Elliptic Paraboloids
   b. Elliptic and Hyperbolic Paraboloids
   c. Hyperboloids of 1-sheet only
   d. Hyperboloids of 2-sheets only
   e. Hyperboloids of 1-sheet or 2-sheets Correct Choice

   \( x^2 - y^2 - z^2 = C \) is a hyperboloid with 2-sheets if \( C > 0 \), and 1-sheet if \( C < 0 \),
   and a cone if \( C = 0 \).

8. Find the plane tangent to the graph of \( z = xe^{xy} \) at the point \( (2, 0) \). Its z-intercept is
   a. 0 Correct Choice
   b. 2
   c. −2
   d. 4
   e. −4

   \( f = xe^{xy} \) \hspace{1cm} \( f(2, 0) = 2 \) \hspace{1cm} \( z = f(2, 0) + f_x(2, 0)(x - 2) + f_y(2, 0)(y - 0) \)
   \( f_x = e^{xy} + xy e^{xy} \) \hspace{1cm} \( f_x(2, 0) = 1 \) \hspace{1cm} \( = 2 + 1(x - 2) + 4(y) = x + 4y \)
   \( f_y = xe^{xy} \) \hspace{1cm} \( f_y(2, 0) = 4 \) \hspace{1cm} The z-intercept is 0.
9. Find the plane tangent to the surface \(xyz + z^2 = 28\) at the point \((4, 3, 2)\).

Its \(z\)-intercept is

a. 0
b. 5 Correct Choice
c. −5
d. 80
e. −80

\[ \vec{N} = \vec{\nabla} F(4, 3, 2) = \langle 6, 8, 16 \rangle \]

\[ \vec{N} \cdot X = \vec{N} \cdot P \]

\[ 6x + 8y + 16z = 6 \cdot 4 + 8 \cdot 3 + 16 \cdot 2 = 80 \]

\[ z = \frac{80}{16} - \frac{6}{16}x - \frac{8}{16}y = 5 - \frac{3}{8}x - \frac{1}{2}y \]

The \(z\)-intercept is 5.

10. Find the line normal to the surface \(xyz + z^2 = 28\) at the point \((4, 3, 2)\).

It intersects the \(xy\)-plane at

a. \((4, 3, 2)\)
b. \((4, 3, 0)\)
c. \(\left(\frac{13}{4}, 2, 0\right)\) Correct Choice
d. \(\left(\frac{19}{4}, 4, 4\right)\)
e. \(\left(\frac{19}{4}, 4, 0\right)\)

\[ \vec{N} = \vec{\nabla} F(4, 3, 2) = \langle 6, 8, 16 \rangle \]

\[ X = P + t \vec{N} \]

\[ (x, y, z) = (4, 3, 2) + t(6, 8, 16) = (4 + 6t, 3 + 8t, 2 + 16t) \]

\[ (x, y, z) \text{ is } z = 0 \text{ or } 2 + 16t = 0 \text{ or } t = -\frac{1}{8} \]

\[ (x, y, z) = \left(\frac{4}{4}, 3 - 1, 2 - 2\right) = \left(\frac{13}{4}, 2, 0\right) \]

11. The salt concentration in a region of sea water is \(\rho = xy^2z^3\). A swimmer is located at \((3, 2, 1)\).

In what direction should the swimmer swim to increase the salt concentration as fast as possible?

a. \(\langle 4, -12, 36 \rangle\)
b. \(\langle -4, 12, -36 \rangle\)
c. \(\langle 4, 12, 36 \rangle\) Correct Choice
d. \(\langle -4, -12, -36 \rangle\)
e. \(\langle 4, -12, -36 \rangle\)

\[ \vec{\nabla} \rho = \langle y^2z^3, 2xyz^3, 3x^2y^2z^2 \rangle = \langle 4, 12, 36 \rangle \]
Work Out: (12 points each. Part credit possible. Show all work.)

Do 4 of the following 5 problems. Cross out the one you do not want graded, here and on page 1.
If you do not specify, #12 will be dropped.

12. Which of the following functions satisfy the Laplace equation $f_{xx} + f_{yy} = 0$?

Show your work!

a. $f = x^2 + y^2$ NO

$$f_{xx} + f_{yy} = 2 + 2 \neq 0$$

b. $f = x^2 - y^2$ YES

$$f_{xx} + f_{yy} = 2 - 2 = 0$$

c. $f = x^3 + 3xy^2$ NO

$$f_{xx} + f_{yy} = 6x + 6x \neq 0$$

d. $f = x^3 - 3xy^2$ YES

$$f_{xx} + f_{yy} = 6x - 6x = 0$$

e. $f = e^{-x}\cos y + e^{-y}\cos x$ YES

$$f_{xx} + f_{yy} = (e^{-x}\cos y - e^{-y}\cos x) + (-e^{-x}\cos y + e^{-y}\cos x) = 0$$

f. $f = e^{-x}\cos y - e^{-y}\cos x$ YES

$$f_{xx} + f_{yy} = (e^{-x}\cos y + e^{-y}\cos x) + (-e^{-x}\cos y - e^{-y}\cos x) = 0$$

13. When two resistors with resistances $R_1$ and $R_2$ are connected in parallel, the net resistance $R$ is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$
or

$$R = \frac{R_1R_2}{R_1 + R_2}.$$  

If $R_1$ and $R_2$ are measured as $R_1 = 2 \pm 0.01$ ohms and $R_2 = 3 \pm 0.04$ ohms, then $R$ can be calculated as $R = \frac{6}{5} \pm \Delta R$ ohms.

Use differentials to estimate the uncertainty $\Delta R$ in the computed value of $R$.

$$\Delta R = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2 = \frac{(R_1 + R_2)(R_2 - R_1)}{(R_1 + R_2)^2} dR_1 + \frac{(R_1 + R_2)(R_1 - R_2)}{(R_1 + R_2)^2} dR_2$$

$$= \frac{(R_2)^2}{(R_1 + R_2)^2} dR_1 + \frac{(R_1)^2}{(R_1 + R_2)^2} dR_2 = \frac{9}{25} (0.01) + \frac{4}{25} (0.04) = \frac{0.09 + 0.16}{25} = 0.01$$
14. The average of a function \( f \) on a curve \( \vec{r}(t) \) is \( f_{\text{ave}} = \frac{\int f ds}{\int ds} \).

Find the average of \( f(x, y) = x^2 \) on the circle \( x^2 + y^2 = 9 \).

**HINTS:** Parametrize the circle. 
\[
\sin^2 A = \frac{1 - \cos(2A)}{2}, \quad \cos^2 A = \frac{1 + \cos(2A)}{2}
\]

\[
\vec{r}(\theta) = (3 \cos \theta, 3 \sin \theta) \quad \vec{v} = (-3 \sin \theta, 3 \cos \theta) \quad |\vec{v}| = \sqrt{9 \sin^2 \theta + 9 \cos^2 \theta} = 3
\]

\[
\int ds = \int_0^{2\pi} 3 \, d\theta = 6\pi \quad f(r(t)) = (3 \cos \theta)^2
\]

\[
\int f ds = \int_0^{2\pi} 9 \cos^2 \theta 3 \, d\theta = 27 \int_0^{2\pi} \frac{1 + \cos(2\theta)}{2} \, d\theta = \frac{27}{2} \left[ \frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right]_0^{2\pi} = 27\pi
\]

\[
f_{\text{ave}} = \frac{27\pi}{6\pi} = \frac{9}{2}
\]

15. A particle moves along the curve \( \vec{r}(t) = (t^3, t^2, t) \) from \( (1, 1, 1) \) to \( (8, 4, 2) \) under the action of the force \( \vec{F} = (z, y, x) \). Find the work done.

\[
\vec{v} = (3t^2, 2t, 1) \quad \vec{F}(\vec{r}(t)) = (t, t^2, t^3)
\]

\[
W = \int_{(1,1,1)}^{(8,4,2)} \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F}(\vec{r}(t)) \cdot \vec{v} \, dt = \int_1^2 (3t^3 + 2t^3 + t^3) \, dt
\]

\[
= \int_1^2 6t^3 \, dt = 6 \left[ t^4 \right]_1^2 = \frac{3}{2} (16 - 1) = \frac{45}{2}
\]

16. The pressure in an ideal gas is given by \( P = k\rho T \) where \( k \) is a constant, \( \rho \) is the density and \( T \) is the temperature. At a certain instant, the measuring instruments are located at \( \vec{r}_o = (1, 2, 3) \) and moving with velocity \( \vec{v} = (4, 5, 6) \) and acceleration \( \vec{a} = (7, 8, 9) \).

At that instant, the density and temperature are measured to be \( \rho = 12 \) and \( T = 300 \) and their gradients are \( \vec{\nabla}\rho = \langle 0.6, 0.4, 0.2 \rangle \) and \( \vec{\nabla}T = \langle 2, 1, 4 \rangle \).

Find \( \frac{dP}{dt} \), the time rate of change of the pressure as seen by the instruments.

**Your answer may depend on \( k \).**

**HINTS:** The pressure, \( P \) is a function of density, \( \rho \), and temperature, \( T \), which are functions of the position coordinates, \( (x, y, z) \), which are functions of time, \( t \). Use the chain rule.

\[
\frac{\partial P}{\partial \rho} = kT = k300 \quad \frac{\partial P}{\partial T} = k \rho = k12
\]

\[
\frac{dP}{dt} = \frac{\partial P}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial P}{\partial T} \frac{dT}{dt} = \frac{\partial P}{\partial \rho} \left( \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \frac{dy}{dt} + \frac{\partial \rho}{\partial z} \frac{dz}{dt} \right) + \frac{\partial P}{\partial T} \left( \frac{\partial T}{\partial x} \frac{dx}{dt} + \frac{\partial T}{\partial y} \frac{dy}{dt} + \frac{\partial T}{\partial z} \frac{dz}{dt} \right)
\]

\[
= \frac{\partial P}{\partial \rho} (\vec{v} \cdot \vec{\nabla}\rho) + \frac{\partial P}{\partial T} (\vec{v} \cdot \vec{\nabla}T) = k300(\langle 4, 5, 6 \rangle \cdot \langle 0.6, 0.4, 0.2 \rangle) + k12(\langle 4, 5, 6 \rangle \cdot \langle 2, 1, 4 \rangle)
\]

\[
= k300(2.4 + 2 + 1.2) + k12(8 + 5 + 24) = k(300 \cdot 5.6 + 12 \cdot 37) = 2124k
\]