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MATH 251 Exam 2 Fall 2006
 Sections 507 Solutions P. Yasskin

1-9	/45	12	/12
10	/12	13	/12
11	/12	14	/12
Total			/105

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_0^z \int_0^{xz} 30x \, dy \, dx \, dz$.

- a. 4
- b. 8
- c. 16
- d. 32
- e. 64 Correct Choice

$$\int_0^2 \int_0^z \int_0^{xz} 30x \, dy \, dx \, dz = \int_0^2 \int_0^z [30xy]_{y=0}^{xz} \, dx \, dz = \int_0^2 \int_0^z 30x^2 z \, dx \, dz = \int_0^2 [10x^3 z]_{x=0}^z \, dz = \int_0^2 10z^4 \, dz$$

$$= [2z^5]_{z=0}^2 = 64$$

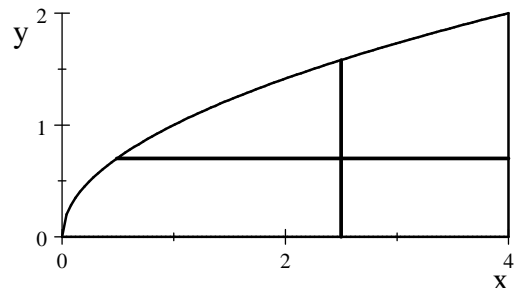
2. Compute $\int_0^2 \int_{y^2}^4 y \sin(x^2) \, dx \, dy$ by interchanging the order of integration.

- a. $\frac{-\cos 16}{2}$
- b. $\frac{1 - \cos 16}{4}$ Correct Choice
- c. $\frac{\cos 16 - 1}{2}$
- d. $\frac{\cos 16}{8}$
- e. $\frac{\cos 16 - 1}{4}$

Plot the region.

In the original order: For each y between 0 and 2, x runs from y^2 to 4.

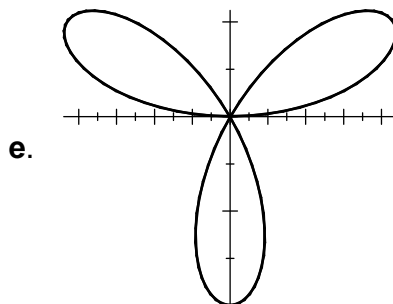
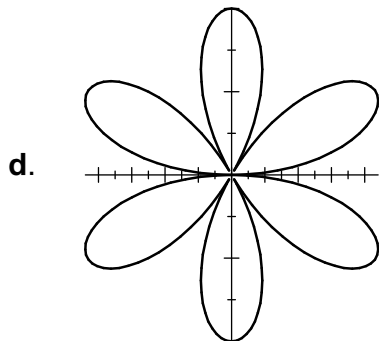
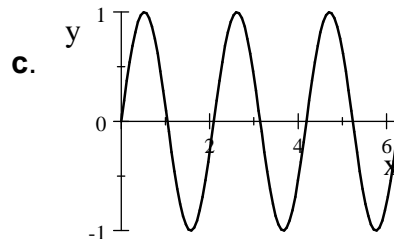
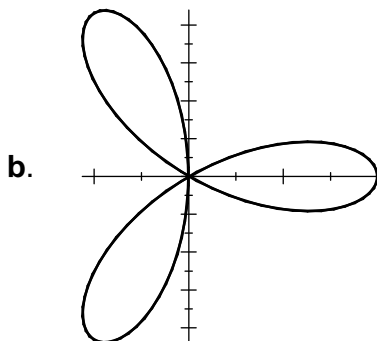
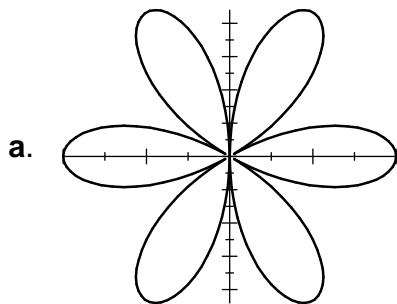
In the reversed order: For each x between 0 and 4, y runs from 0 to \sqrt{x} .



$$\int_0^2 \int_{y^2}^4 y \sin(x^2) \, dx \, dy = \int_0^4 \int_0^{\sqrt{x}} y \sin(x^2) \, dy \, dx = \int_0^4 \left[\frac{y^2}{2} \sin(x^2) \right]_0^{\sqrt{x}} \, dx = \frac{1}{2} \int_0^4 x \sin(x^2) \, dx$$

$$= \left[\frac{-\cos(x^2)}{4} \right]_0^4 = \frac{1 - \cos 16}{4}$$

3. Which of the following is the polar plot of $r = \sin(3\theta)$?

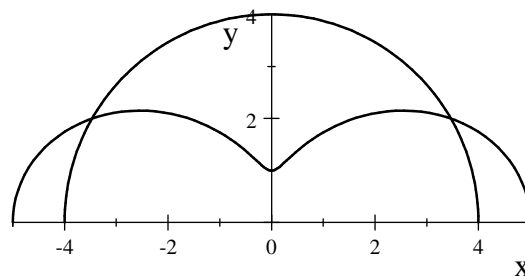


Correct Choice

(c) is the rectangular plot of $r = \sin(3\theta)$. (e) is its polar plot because there are 3 positive loops and 3 negative loops which retrace the positive loops.

4. Find the area of the region inside the circle $r = 4$ outside the polar curve $r = 3 + 2\cos(2\theta)$ with $y \geq 0$.

The area is given by the integral:



a. $A = \int_{\pi/3}^{5\pi/3} \int_{3+2\cos(2\theta)}^4 r dr d\theta$

b. $A = \int_{\pi/3}^{5\pi/3} \int_{3+2\cos(2\theta)}^4 r dr d\theta$

c. $A = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^4 r dr d\theta$

d. $A = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^4 r dr d\theta$ Correct Choice

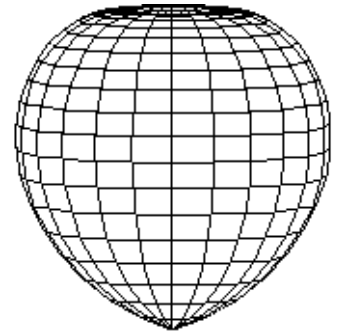
e. $A = \int_{\pi/3}^{5\pi/3} \int_4^{3+2\cos(2\theta)} r dr d\theta$

$r = 3 + 2\cos(2\theta) = 4 \Rightarrow \cos(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$ by symmetry

$A = \iint dA = \int_{\pi/6}^{5\pi/6} \int_{3+2\cos(2\theta)}^4 r dr d\theta$

5. Find the volume of the apple given in spherical coordinates by $\rho = \varphi$.

The volume is given by the integral:



- a. $\frac{2\pi}{3} \int_0^\pi \varphi^3 \sin \varphi d\varphi$ Correct Choice
- b. $2\pi \int_0^{2\pi} \varphi^2 \sin \varphi d\varphi$
- c. $2\pi \int_0^\pi \varphi^2 \sin \varphi d\varphi$
- d. $\pi \int_0^{2\pi} \varphi^2 \sin \varphi d\varphi$
- e. $\pi \int_0^\pi \varphi^2 \sin \varphi d\varphi$

$$V = \iiint dV = \int_0^{2\pi} \int_0^\pi \int_0^\varphi \rho^2 \sin \varphi d\rho d\varphi d\theta = 2\pi \int_0^\pi \left[\frac{\rho^3}{3} \right]_{\rho=0}^\varphi \sin \varphi d\varphi = \frac{2\pi}{3} \int_0^\pi \varphi^3 \sin \varphi d\varphi$$

6. Find a scalar potential f for the vector field $\vec{F} = (y + z, x + z, x + y + 2z)$.

Then evaluate $f(1, 1, 1) - f(0, 0, 0)$:

- a. 1
- b. 2
- c. 4 Correct Choice
- d. 5
- e. 7

$$\partial_x f = y + z \quad f = xy + xz + g(y, z)$$

$$\partial_y f = x + z \quad x + \partial_y g = x + z \quad \partial_y g = z \quad g = yz + h(z) \quad f = xy + xz + yz + h(z)$$

$$\partial_z f = x + y + 2z \quad x + y + \frac{dh}{dz} = x + y + 2z \quad \frac{dh}{dz} = 2z \quad h = z^2 \quad f = xy + xz + yz + z^2$$

$$f(1, 1, 1) - f(0, 0, 0) = (1 + 1 + 1 + 1) - (0 + 0 + 0 + 0) = 4$$

7. Which vector field cannot be written as $\vec{\nabla} \times \vec{F}$ for any vector field \vec{F} .

a. $\vec{A} = (x, y, -2z)$

$$\vec{\nabla} \cdot \vec{A} = 1 + 1 - 2 = 0$$

b. $\vec{B} = (xz, yz, z^2)$ Correct Choice

$$\vec{\nabla} \cdot \vec{B} = z + z + 2z = 4z \neq 0$$

c. $\vec{C} = (xz, yz, -z^2)$

$$\vec{\nabla} \cdot \vec{C} = z + z - 2z = 0$$

d. $\vec{D} = (z \sin x, -yz \cos x, y \sin x)$

$$\vec{\nabla} \cdot \vec{D} = z \cos x - z \cos x = 0$$

e. $\vec{E} = (x \sin y, \cos y, x \cos y)$

$$\vec{\nabla} \cdot \vec{E} = \sin y - \sin y = 0$$

Since it is always true that $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$, \vec{B} cannot be $\vec{B} = \vec{\nabla} \times \vec{F}$.

On the other hand, \vec{A} , \vec{C} , \vec{D} and \vec{E} probably have vector potentials.

8. Find the total mass of a plate occupying the region between $y = x^2$ and $y = 4$ if the mass density is $\rho = y$.

a. $\frac{32}{3}$

b. $\frac{32}{9}$

c. $\frac{64}{5}$

d. $\frac{128}{5}$ Correct Choice

e. $\frac{256}{5}$

$$y = x^2 = 4 \quad x = \pm 2$$

$$M = \iint \rho dA = \int_{-2}^2 \int_{x^2}^4 y dy dx = \int_{-2}^2 \left[\frac{y^2}{2} \right]_{y=x^2}^4 dx = \frac{1}{2} \int_{-2}^2 (16 - x^4) dx = \frac{1}{2} \left[16x - \frac{x^5}{5} \right]_{x=-2}^2 = \frac{128}{5}$$

9. Find the center of mass of a plate occupying the region between $y = x^2$ and $y = 4$ if the mass density is $\rho = y$.

a. $(0, \frac{20}{7})$ Correct Choice

b. $(0, \frac{12}{5})$

c. $(0, \frac{512}{7})$

d. $(0, \frac{128}{5})$

e. $(0, \frac{14}{5})$

$$M = \frac{128}{5} \quad \text{By symmetry, } \bar{x} = 0.$$

$$\begin{aligned} y\text{-mom} = M_x &= \iint y \rho dA = \int_{-2}^2 \int_{x^2}^4 y^2 dy dx = \int_{-2}^2 \left[\frac{y^3}{3} \right]_{y=x^2}^4 dx = \frac{1}{3} \int_{-2}^2 (64 - x^6) dx = \frac{1}{3} \left[64x - \frac{x^7}{7} \right]_{x=-2}^2 \\ &= \frac{2}{3} \left(128 - \frac{128}{7} \right) = \frac{512}{7} \quad \bar{y} = \frac{y\text{-mom}}{M} = \frac{M_x}{M} = \frac{512}{7} \frac{5}{128} = \frac{20}{7} \end{aligned}$$

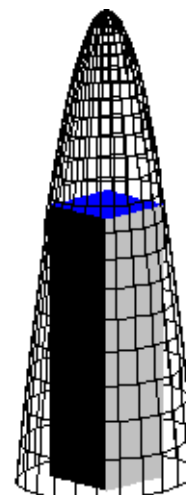
Work Out: (12 points each. Part credit possible. Show all work.)

10. Find the dimensions and volume of the largest box which sits on the xy -plane and whose upper vertices are on the elliptic paraboloid $z = 12 - 2x^2 - 3y^2$.

You do not need to show it is a maximum.

You MUST eliminate the constraint.

Do not use Lagrange multipliers.



$$\text{Maximize } V = (2x)(2y)z = 4xy(12 - 2x^2 - 3y^2) = 48xy - 8x^3y - 12xy^3$$

$$V_x = 48y - 24x^2y - 12y^3 = -12y(2x^2 + y^2 - 4) = 0$$

$$V_y = 48x - 8x^3 - 36xy^2 = -4x(2x^2 + 9y^2 - 12) = 0$$

Since $x = 0$ or $y = 0$ gives $V = 0$, we can assume $x \neq 0$ and $y \neq 0$.

$$2x^2 + y^2 = 4 \quad \text{and} \quad 2x^2 + 9y^2 = 12$$

$$\text{Subtract: } 8y^2 = 8 \Rightarrow y = 1$$

$$\text{Substitute back: } 2x^2 + 1 = 4 \Rightarrow x^2 = \frac{3}{2} \Rightarrow x = \sqrt{\frac{3}{2}}$$

$$\text{Substitute back: } z = 12 - 2x^2 - 3y^2 = 12 - 2\left(\frac{3}{2}\right) - 3(1) = 6$$

$$\text{The dimensions are: } L = 2x = \sqrt{6} \quad W = 2y = 2 \quad H = z = 6$$

$$\text{The volume is: } V = LWH = \sqrt{6}(2)(6) = 12\sqrt{6}$$

11. A pot of water is sitting on a stove. The pot is a cylinder of radius 3 inches and height 4 inches.

If the origin is located at the center of the bottom, then the temperature of the water is

$$T = 102 + x^2 + y^2 - z. \quad \text{Find the average temperature of the water: } T_{\text{ave}} = \frac{\iiint T dV}{\iiint dV}.$$

$$\iiint dV = \int_0^{2\pi} \int_0^3 \int_0^4 r dz dr d\theta = [\theta]_0^{2\pi} \left[\frac{r^2}{2} \right]_0^3 [z]_0^4 = 2\pi \left(\frac{9}{2} \right) (4) = 36\pi \quad (= \pi r^2 h)$$

$$\begin{aligned} \iiint T dV &= \int_0^{2\pi} \int_0^3 \int_0^4 (102 + r^2 - z) r dz dr d\theta = 2\pi \int_0^3 \left[102z + r^2z - \frac{z^2}{2} \right]_{z=0}^4 r dr \\ &= 2\pi \int_0^3 (408 + 4r^2 - 8) r dr = 2\pi \int_0^3 (400r + 4r^3) dr = 2\pi [200r^2 + r^4]_{r=0}^3 = 2\pi(1881) = 3762\pi \end{aligned}$$

$$T_{\text{ave}} = \frac{3762\pi}{36\pi} = \frac{209}{2}$$

12. Compute $\iint_D x dx dy$ over the "diamond"

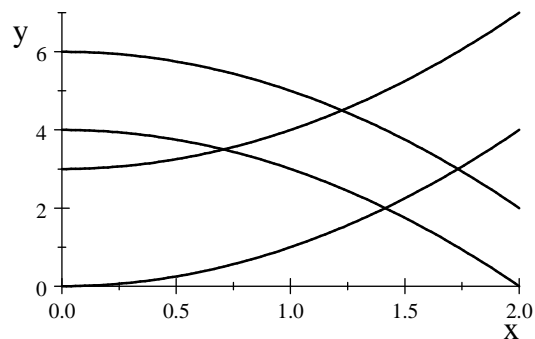
shaped region bounded by the curves

$$y = x^2 \quad y = 3 + x^2 \quad y = 4 - x^2 \quad y = 6 - x^2$$

Use the curvilinear coordinates

$$u = y + x^2 \quad \text{and} \quad v = y - x^2.$$

(Half credit for using rectangular coordinates.)



The boundary curves are $v = 0$, $v = 3$, $u = 4$, $u = 6$

We solve for x and y and compute the Jacobian factor:

$$u + v = 2y \quad u - v = 2x^2 \quad x = \frac{\sqrt{u - v}}{\sqrt{2}} \quad y = \frac{u + v}{2}$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{(u - v)^{-1/2}}{2\sqrt{2}} & \frac{1}{2} \\ -\frac{(u - v)^{-1/2}}{2\sqrt{2}} & \frac{1}{2} \end{vmatrix} = \left| \frac{(u - v)^{-1/2}}{4\sqrt{2}} + \frac{(u - v)^{-1/2}}{4\sqrt{2}} \right| = \frac{(u - v)^{-1/2}}{2\sqrt{2}}$$

The integrand is $x = \frac{\sqrt{u - v}}{\sqrt{2}}$. So

$$\iint_D x dx dy = \int_0^3 \int_4^6 \frac{\sqrt{u - v}}{\sqrt{2}} \frac{(u - v)^{-1/2}}{2\sqrt{2}} du dv = \int_0^3 \int_4^6 \frac{1}{4} du dv = \frac{1}{4}(3)(2) = \frac{3}{2}$$

13. The sides of a cylinder C of radius 3 and height 4 may be parametrized by

$$R(h, \theta) = (3 \cos \theta, 3 \sin \theta, h) \quad \text{for} \quad 0 \leq \theta \leq 2\pi \quad \text{and} \quad 0 \leq h \leq 4.$$

Compute $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ for $\vec{F} = (-yz^2, xz^2, z^3)$ and outward normal.

HINT: Find \vec{e}_h , \vec{e}_θ , $\vec{N} = \vec{e}_h \times \vec{e}_\theta$, $\vec{\nabla} \times \vec{F}$ and $(\vec{\nabla} \times \vec{F})(\vec{R}(h, \theta))$.

$$R(h, \theta) = (3 \cos \theta, 3 \sin \theta, h)$$

$$\vec{e}_h = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \end{pmatrix}$$

$$\vec{e}_\theta = \begin{pmatrix} -3 \sin \theta & 3 \cos \theta & 0 \end{pmatrix}$$

$$\vec{N} = \vec{e}_h \times \vec{e}_\theta = \hat{i}(-3 \cos \theta) - \hat{j}(3 \sin \theta) + \hat{k}(0) = (-3 \cos \theta, -3 \sin \theta, 0)$$

$$\text{Reverse } \vec{N} = (3 \cos \theta, 3 \sin \theta, 0)$$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{i}(0 - 2xz) - \hat{j}(0 - 2yz) + \hat{k}(z^2 - -z^2) = (-2xz, -2yz, 2z^2)$$

$$(\vec{\nabla} \times \vec{F})(\vec{R}(h, \theta)) = (-6h \cos \theta, -6h \sin \theta, 2h^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = -18h \cos^2 \theta - 18h \sin^2 \theta = -18h$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint \vec{\nabla} \times \vec{F} \cdot \vec{N} dh d\theta = \int_0^{2\pi} \int_0^4 -18h dh d\theta = -[2\pi 9h^2]_0^4 = -288\pi$$

14. The hemispherical surface $x^2 + y^2 + z^2 = 9$ has surface density $\rho = x^2 + y^2$.

The surface may be parametrized by $\vec{R}(\varphi, \theta) = (3 \sin \varphi \cos \theta, 3 \sin \varphi \sin \theta, 3 \cos \varphi)$.

Find the mass and center of mass of the surface.

HINT: Find \vec{e}_φ , \vec{e}_θ , $\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta$, $|\vec{N}|$ and $\rho(\vec{R}(\varphi, \theta))$.

$$\vec{R}(\varphi, \theta) = (3 \sin \varphi \cos \theta, 3 \sin \varphi \sin \theta, 3 \cos \varphi)$$

$$\vec{e}_\varphi = \begin{pmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 \cos \varphi \cos \theta & 3 \cos \varphi \sin \theta & -3 \sin \varphi \end{pmatrix}$$

$$\vec{e}_\theta = \begin{pmatrix} -3 \sin \varphi \sin \theta & 3 \sin \varphi \cos \theta & 0 \end{pmatrix}$$

$$\vec{N} = \vec{e}_\varphi \times \vec{e}_\theta = \hat{i}(9 \sin^2 \varphi \cos \theta) - \hat{j}(-9 \sin^2 \varphi \sin \theta) + \hat{k}(9 \sin \varphi \cos \varphi \cos^2 \theta + 9 \sin \varphi \cos \varphi \sin^2 \theta)$$

$$\vec{N} = (9 \sin^2 \varphi \cos \theta, 9 \sin^2 \varphi \sin \theta, 9 \sin \varphi \cos \varphi)$$

$$|\vec{N}| = \sqrt{(9 \sin^2 \varphi \cos \theta)^2 + (9 \sin^2 \varphi \sin \theta)^2 + (9 \sin \varphi \cos \varphi)^2} = 9 \sqrt{\sin^4 \varphi + \sin^2 \varphi \cos^2 \varphi} = 9 \sin \varphi$$

$$\rho(\vec{R}(\varphi, \theta)) = (3 \sin \varphi \cos \theta)^2 + (3 \sin \varphi \sin \theta)^2 = 9 \sin^2 \varphi$$

$$M = \iint \rho dS = \iint \rho(\vec{R}(\varphi, \theta)) |\vec{N}| d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/2} 9 \sin^2 \varphi 9 \sin \varphi d\varphi d\theta = 2\pi 81 \int_0^{\pi/2} (1 - \cos^2 \varphi) \sin \varphi d\varphi$$

Let $u = \cos \varphi$ $du = -\sin \varphi d\varphi$

$$M = -162\pi \int_1^0 (1 - u^2) du = -162\pi \left[u - \frac{u^3}{3} \right]_1^0 = 162\pi \left(1 - \frac{1}{3} \right) = \frac{324\pi}{3} = 108\pi$$

By symmetry $\bar{x} = \bar{y} = 0$

$$\begin{aligned} z\text{-mom} = M_{xy} &= \iint z \rho dS = \iint 3 \cos \varphi \rho(\vec{R}(\varphi, \theta)) |\vec{N}| d\varphi d\theta = \int_0^{2\pi} \int_0^{\pi/2} 3 \cos \varphi 9 \sin^2 \varphi 9 \sin \varphi d\varphi d\theta \\ &= 2\pi 243 \int_0^{\pi/2} \cos \varphi \sin^3 \varphi d\varphi = 486\pi \left[\frac{\sin^4 \varphi}{4} \right]_0^{\pi/2} = \frac{243\pi}{2} \end{aligned}$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{243\pi}{2} \frac{1}{108\pi} = \frac{9}{8}$$