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11	/15
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MATH 251 Final Exam Fall 2006

Sections 507 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. For the curve $\vec{r}(t) = (t\cos t, t\sin t)$, which of the following is false?

a. The velocity is $\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$

b. The speed is $|\vec{v}| = \sqrt{1+t^2}$

c. The acceleration is $\vec{a} = (-2\sin t - t\cos t, 2\cos t - t\sin t)$

d. The arclength between $t = 0$ and $t = 1$ is $L = \int_0^1 t\sqrt{1+t^2}\,dt$ Correct Choice

e. The tangential acceleration is $a_T = \dfrac{t}{\sqrt{1+t^2}}$

$$\vec{v} = (\cos t - t\sin t, \sin t + t\cos t)$$

$$|\vec{v}| = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{\cos^2 t + t^2\cos^2 t + \sin^2 t + t^2\sin^2 t} = \sqrt{1+t^2}$$

$$\vec{a} = (-2\sin t - t\cos t, 2\cos t - t\sin t)$$

$$L = \int_0^1 |\vec{v}|\,dt = \int_0^1 \sqrt{1+t^2}\,dt$$

$$a_T = \frac{d|\vec{v}|}{dt} = \frac{2t}{2\sqrt{1+t^2}} \quad \text{or}$$

$$a_T = \vec{a}\cdot\hat{T} = (-2\sin t - t\cos t, 2\cos t - t\sin t)\cdot\frac{1}{\sqrt{1+t^2}}(\cos t - t\sin t, \sin t + t\cos t)$$

$$= \frac{1}{\sqrt{1+t^2}}[(-2\sin t - t\cos t)(\cos t - t\sin t) + (2\cos t - t\sin t)(\sin t + t\cos t)] = \frac{t}{\sqrt{1+t^2}}$$

2. Find the plane tangent to the surface $x^2z^2 + y^4 = 5$ at the point $(2,1,1)$.

a. $2x + y + z = 6$

b. $2x + y + z = 5$

c. $x + y + 2z = 5$ Correct Choice

d. $x - y + 2z = 3$

e. $x - y + 2z = 6$

$$f = x^2z^2 + y^4 \qquad P = (2,1,1) \qquad \vec{\nabla}f = (2xz^2, 4y^3, 2x^2z) \qquad \vec{N} = \vec{\nabla}f\big|_P = (4,4,8)$$

$$\vec{N}\cdot X = \vec{N}\cdot P \qquad 4x + 4y + 8z = 8 + 4 + 8 = 20 \qquad x + y + 2z = 5$$

3. Let $L = \lim_{(x,y)\to(0,0)} \frac{x^2 + xy^2}{x^2 + y^4}$

- **a.** L exists and $L = 1$ by looking at the paths $y = mx$.
- **b.** L does not exist by looking at the paths $y = x$ and $y = \sqrt{x}$.
- **c.** L does not exist by looking at the paths $y = \sqrt{x}$ and $y = -\sqrt{x}$.
- **d.** L does not exist by looking at the paths $x = my^2$. Correct Choice
- **e.** L does not exist by looking at the paths $x = y^3$ and $x = -y^3$.

Along $y = mx$, we have $L = \lim_{x\to 0} \frac{x^2 + m^2x^4}{x^2 + m^4x^4} = \lim_{x\to 0} \frac{1 + m^2x^3}{1 + m^4x^2} = 1$, which proves nothing.

Along $y = \pm\sqrt{x}$, we have $L = \lim_{x\to 0} \frac{x^2 + x^2}{x^2 + x^2} = 1$, which proves nothing.

Along $x = my^2$, we have $L = \lim_{y\to 0} \frac{m^2y^4 + my^4}{m^4y^4 + y^4} = \lim_{y\to 0} \frac{m^2 + m}{m^4 + 1}$, which depends on m and proves the limit does not exist.

Along $x = \pm y^3$, we have $L = \lim_{y\to 0} \frac{y^6 \pm y^5}{y^6 + y^4} = \lim_{y\to 0} \frac{y^2 \pm y}{y^2 + 1} = 0$, which (by itself) proves nothing.

4. The point $(1,-3)$ is a critical point of the function $f = xy^2 - 3x^3 + 6y$. It is a

- **a.** local minimum.
- **b.** local maximum.
- **c.** saddle point. Correct Choice
- **d.** inflection point.
- **e.** The Second Derivative Test fails.

$f_x = y^2 - 9x^2 \qquad f_y = 2xy + 6 \qquad f_{xx} = -18x \qquad f_{yy} = 2x \qquad f_{xy} = 2y$

$f_{xx}(1,-3) = -18 \qquad f_{yy}(1,-3) = 2 \qquad f_{xy}(1,-3) = -6 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = -36 - 36 = -72$

saddle point

5. The dimensions of a closed rectangular box are measured as 70 cm, 50 cm and 40 cm with a possible error of 0.2 cm in each dimension. Use differentials to estimate the maximum error in the calculated surface area of the box.

- **a.** 8
- **b.** 16
- **c.** 32
- **d.** 64
- **e.** 128 Correct Choice

$A = 2xy + 2xz + 2yz \qquad dS = \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz$

$dA = 2(90)(.2) + 2(110)(.2) + 2(120)(.2) = .4(320) = 128$

6. Consider the quarter of the cylinder $x^2+y^2 \le 4$ with $x \ge 0$, $y \ge 0$ and $0 \le z \le 8$.
Find the total mass of the quarter cylinder if the density is $\rho = e^{x^2+y^2}$.

a. $2\pi(e^4-1)$ Correct Choice

b. $8\pi(e^4-1)$

c. $2\pi e^4$

d. $8\pi e^4$

e. 4

$$M = \iiint \rho\, dV = \int_0^8 \int_0^{\pi/2} \int_0^2 e^{r^2} r\, dr\, d\theta\, dz = (8)\left(\frac{\pi}{2}\right)\left[\frac{e^{r^2}}{2}\right]_0^2 = 2\pi(e^4-1)$$

7. Consider the quarter of the cylinder $x^2+y^2 \le 4$ with $x \ge 0$, $y \ge 0$ and $0 \le z \le 8$.
Find the z- component of the center of mass of the quarter cylinder if the density is $\rho = e^{x^2+y^2}$.

a. $2\pi(e^4-1)$

b. $8\pi(e^4-1)$

c. $2\pi e^4$

d. $8\pi e^4$

e. 4 Correct Choice

$$z\text{-mom} = \iiint z\rho\, dV = \int_0^8 \int_0^{\pi/2} \int_0^2 z e^{r^2} r\, dr\, d\theta\, dz = \left[\frac{z^2}{2}\right]_0^8 \left(\frac{\pi}{2}\right)\left[\frac{e^{r^2}}{2}\right]_0^2 = 8\pi(e^4-1)$$

$$\bar{z} = \frac{z\text{-mom}}{M} = \frac{8\pi(e^4-1)}{2\pi(e^4-1)} = 4$$ OR by symmetry, $\bar{z}$ must be half way up.

8. Compute the line integral $\int y\,dx - x\,dy$ counterclockwise around the semicircle $x^2+y^2 = 9$ from $(3,0)$ to $(-3,0)$. (HINT: Parametrize the curve.)

a. -9π Correct Choice

b. -3π

c. π

d. 3π

e. 9π

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ $\quad \vec{v} = (-3\sin\theta, 3\cos\theta)$ Oriented correctly.

$\vec{F} = (y,-x) = (3\sin\theta, -3\cos\theta)$ $\quad \vec{F}\cdot\vec{v} = -9\sin^2\theta - 9\cos^2\theta = -9$

$\int y\,dx - x\,dy = \int \vec{F}\cdot d\vec{s} = \int \vec{F}\cdot\vec{v}\,d\theta = \int_0^\pi -9\,d\theta = -9\pi$

9. Compute the line integral $\int \vec{F} \cdot d\vec{s}$ for the vector field $\vec{F} = (y, x)$ along the curve $\vec{r}(t) = \left(e^{\cos(t^2)}, e^{\sin(t^2)}\right)$ for $0 \le t \le \sqrt{\pi}$. (HINT: Find a scalar potential.)

a. $e - \frac{1}{e}$

b. $\frac{1}{e} - e$ Correct Choice

c. $\frac{2}{e}$

d. $2e$

e. 0

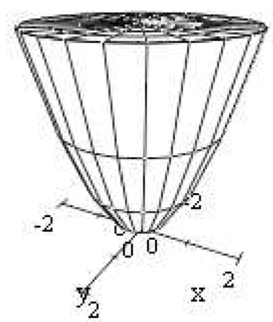
$\vec{F} = \vec{\nabla} f$ for $f = xy$ $A = \vec{r}(0) = (e^{\cos 0}, e^{\sin 0}) = (e, 1)$ $B = \vec{r}(\sqrt{\pi}) = (e^{\cos\pi}, e^{\sin\pi}) = (e^{-1}, 1)$

By the F.T.C.C. $\int_A^B \vec{F} \cdot d\vec{s} = \int_A^B \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = f(e^{-1}, 1) - f(e, 1) = (e^{-1} \cdot 1) - (e \cdot 1) = \frac{1}{e} - e$

10. Consider the parabolic surface P given by $z = x^2 + y^2$ for $z \le 4$ with normal pointing up and in, the disk D given by $x^2 + y^2 \le 4$ and $z = 4$ with normal pointing up, and the volume V between them. Given that for a certain vector field $\vec{F}$ we have

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = 14 \quad \text{and} \quad \iint_D \vec{F} \cdot d\vec{S} = 3$$

compute $\iint_P \vec{F} \cdot d\vec{S}$.



a. 17

b. 11

c. 8

d. -11 Correct Choice

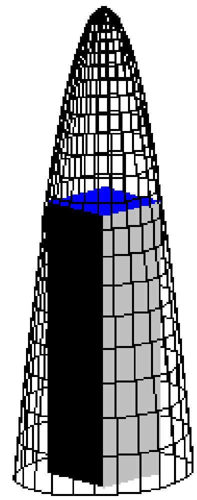
e. -17

By Gauss' Theorem: $\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_D \vec{F} \cdot d\vec{S} - \iint_P \vec{F} \cdot d\vec{S}$

The minus sign reverses the orientation of P to point outward. Thus

$$\iint_P \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot d\vec{S} - \iiint_V \vec{\nabla} \cdot F\, dV = 3 - 14 = -11$$

Work Out: (15 points each. Part credit possible.)



11. Find the dimensions and volume of the largest box which sits on the xy-plane and whose upper vertices are on the elliptic paraboloid $z + 2x^2 + 3y^2 = 12$.

You do not need to show it is a maximum.
You MUST use the Method of Lagrange multipliers.

Half credit for the Method of Elminating the Constraint.

Maximize $V = LWH = (2x)(2y)z = 4xyz$ subject to the constraint $g = z + 2x^2 + 3y^2 = 12$.

$\vec{\nabla} V = (4yz, 4xz, 4xy) \qquad \vec{\nabla} g = (4x, 6y, 1)$

$\vec{\nabla} V = \lambda \vec{\nabla} g \qquad \Rightarrow \qquad 4yz = \lambda 4x \qquad 4xz = \lambda 6y \qquad 4xy = \lambda$

$\lambda = 4xy \qquad \Rightarrow \qquad 4yz = 16x^2 y \qquad 4xz = 24xy^2$

Since $V \neq 0$, we can assume $x \neq 0$ and $y \neq 0$ and $z \neq 0$.

So $z = 4x^2 \qquad z = 6y^2 \qquad 2x^2 = 3y^2$

The constraint becomes: $4x^2 + 2x^2 + 2x^2 = 12$ or $8x^2 = 12$

$x = \sqrt{\dfrac{3}{2}} \qquad y = \sqrt{\dfrac{2}{3}}\, x = 1 \qquad z = 4x^2 = 6$

The dimensions are: $L = 2x = \sqrt{6} \qquad W = 2y = 2 \qquad H = z = 6$

The volume is: $V = LWH = \sqrt{6}(2)(6) = 12\sqrt{6}$

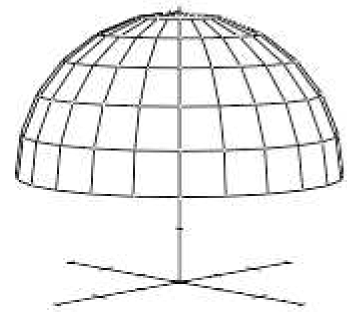
12. The hemisphere H given by

$$x^2+y^2+(z-2)^2=9 \quad \text{for} \quad z\ge 2$$

has center $(0,0,2)$ and radius 3. Verify Stokes' Theorem

$$\iint_H \vec\nabla\times\vec F\cdot d\vec S=\oint_{\partial H}\vec F\cdot d\vec s$$

for this hemisphere H with normal pointing up and out and the vector field $\vec F=(yz,-xz,z)$.



Be sure to check and explain the orientations. Use the following steps:

a. The hemisphere may be parametrized by

$\vec R(\theta,\varphi)=(3\sin\varphi\cos\theta,3\sin\varphi\sin\theta,2+3\cos\varphi)$

Compute the surface integral by successively finding:

$\vec e_\theta,\ \ \vec e_\varphi,\ \ \vec N,\ \ \vec\nabla\times\vec F,\ \ \vec\nabla\times\vec F\big(\vec R(\theta,\varphi)\big),\ \ \iint_H\vec\nabla\times\vec F\cdot d\vec S$

$$\begin{array}{r|ccc|} & \hat\imath & \hat\jmath & \hat k \\ \vec e_\theta = & (-3\sin\varphi\sin\theta, & 3\sin\varphi\cos\theta, & 0\ \ ) \\ \vec e_\varphi = & (3\cos\varphi\cos\theta, & 3\cos\varphi\sin\theta, & -3\sin\varphi) \end{array}$$

$$\vec N=\vec e_\theta\times\vec e_\varphi=\hat\imath(-9\sin^2\varphi\cos\theta)-\hat\jmath(9\sin^2\varphi\sin\theta)+\hat k(-9\sin\varphi\cos\varphi\sin^2\theta-9\sin\varphi\cos\varphi\cos^2\theta)$$
$$=(-9\sin^2\varphi\cos\theta,-9\sin^2\varphi\sin\theta,-9\sin\varphi\cos\varphi)$$

$\vec N$ points down and in. Reverse it: $\vec N=(9\sin^2\varphi\cos\theta,9\sin^2\varphi\sin\theta,9\sin\varphi\cos\varphi)$

$$\vec\nabla\times\vec F=\begin{vmatrix}\hat\imath & \hat\jmath & \hat k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ yz, & -xz, & z\end{vmatrix}=\hat\imath(0--x)-\hat\jmath(0-y)+\hat k(-z-z)=(x,y,-2z)$$

$$\vec\nabla\times\vec F\big(\vec R(\theta,\varphi)\big)=(3\sin\varphi\cos\theta,3\sin\varphi\sin\theta,-2(2+3\cos\varphi))$$

$$\vec\nabla\times\vec F\cdot\vec N=27\sin^3\varphi\cos^2\theta+27\sin^3\varphi\sin^2\theta-18\sin\varphi\cos\varphi(2+3\cos\varphi)$$
$$=27\sin^3\varphi-36\sin\varphi\cos\varphi-54\sin\varphi\cos^2\varphi$$

$$\iint_H\vec\nabla\times\vec F\cdot d\vec S=\iint_H\vec\nabla\times\vec F\cdot\vec N\,d\theta\,d\varphi=\int_0^{\pi/2}\int_0^{2\pi}(27\sin^3\varphi-36\sin\varphi\cos\varphi-54\sin\varphi\cos^2\varphi)\,d\theta\,d\varphi$$

$$=2\pi\int_0^{\pi/2}(27(1-\cos^2\varphi)\sin\varphi-36\sin\varphi\cos\varphi-54\sin\varphi\cos^2\varphi)\,d\varphi \qquad \text{Let } u=\cos\varphi.$$

$$=2\pi\left[-27\left(\cos\varphi-\frac{\cos^3\varphi}{3}\right)+18\cos^2\varphi+18\cos^3\varphi\right]_0^{\pi/2}=-2\pi\left(-27\left(1-\frac{1}{3}\right)+18+18\right)$$

$$=-36\pi$$

Problem Continued

b. Parametrize the boundary circle ∂H and compute the line integral by successively finding:

$\vec{r}(\theta)$, $\vec{v}(\theta)$, $\vec{F}(\vec{r}(\theta))$, $\oint_{\partial H} \vec{F} \cdot d\vec{s}$. Recall: $\vec{F} = (yz, -xz, z)$

$$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 2)$$

$$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$$

By the right hand rule the upper curve must be traversed counterclockwise which $\vec{v}$ does.

$$\vec{F}(\vec{r}(\theta)) = (6\sin\theta, -6\cos\theta, 2)$$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} -18\sin^2\theta - 18\cos^2\theta\, d\theta = \int_0^{2\pi} -18\, d\theta = -36\pi$$

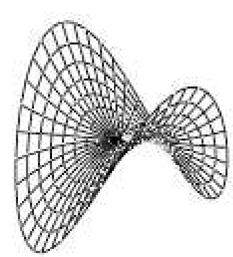
They agree!

13. The spider web at the right is the graph of the hyperbolic paraboloid $z = xy$.

It may be parametrized as

$$\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r^2\sin\theta\cos\theta).$$

Find the area of the web for $r \le \sqrt{3}$.



$$\vec{R}_r = (\cos\theta, \quad \sin\theta, \quad 2r\sin\theta\cos\theta)$$

$$\vec{R}_\theta = (-r\sin\theta, r\cos\theta, r^2(\cos^2\theta - \sin^2\theta))$$

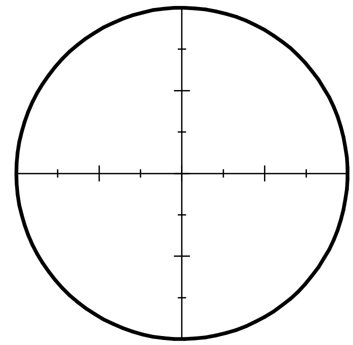
$$\vec{N} = i(r^2\sin\theta(\cos^2\theta - \sin^2\theta) - 2r^2\sin\theta\cos^2\theta) - j(r^2\cos\theta(\cos^2\theta - \sin^2\theta) - -2r^2\sin^2\theta\cos\theta)$$
$$+ k(r\cos^2\theta - -r\sin^2\theta) = (-r^2\sin\theta\cos^2\theta - r^2\sin^3\theta, -r^2\cos^3\theta - r^2\sin^2\theta\cos\theta, r)$$
$$= (-r^2\sin\theta, -r^2\cos\theta, r)$$

$$\left|\vec{N}\right| = \sqrt{r^4\sin^2\theta + r^4\cos^2\theta + r^2}\,\sqrt{r^4 + r^2} = r\sqrt{r^2+1}$$

$$A = \int_0^{2\pi}\int_0^{\sqrt{3}} \left|\vec{N}\right| dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} r(r^2+1)^{1/2}\, dr\, d\theta = 2\pi\left[\frac{2}{3}\frac{(r^2+1)^{3/2}}{2}\right]_0^{\sqrt{3}}$$
$$= \frac{2\pi}{3}\left[(4)^{3/2} - (1)^{3/2}\right] = \frac{2\pi}{3}[8-1] = \frac{14\pi}{3}$$

14. Green's Theorem states:

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \oint_{\partial R} P\,dx + Q\,dy$$



Verify Green's Theorem for the functions

$P = -x^2 y$ and $Q = xy^2$

on the region inside the circle $x^2 + y^2 = 16$.

Use the following steps:

a. Compute $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y}$.

Then compute $\iint_R \left(\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} \right) dx\,dy$ by converting to polar coordinates.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = y^2 - -x^2 = x^2 + y^2 = r^2 \qquad dx\,dy = r\,dr\,d\theta$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \int_0^{2\pi} \int_0^4 r^2 r\,dr\,d\theta = 2\pi \left[\frac{r^4}{4} \right]_{r=0}^{4} = 128\pi$$

b. Parametrize the boundary circle.

Compute P, Q, dx and dy on the boundary curve.

Then compute $\oint_{\partial R} P\,dx + Q\,dy$ around the boundary.

The boundary circle $x^2 + y^2 = 16$ may be parametrized by $\vec{r}(\theta) = (4\cos\theta, 4\sin\theta)$.

$P = -x^2 y = -64\cos^2\theta \sin\theta \qquad Q = xy^2 = 64\cos\theta\sin^2\theta$

$dx = -4\sin\theta\,d\theta \qquad dy = 4\cos\theta\,d\theta$

$P\,dx + Q\,dy = 256\cos^2\theta\sin^2\theta\,d\theta + 256\cos^2\theta\sin^2\theta\,d\theta = 512\cos^2\theta\sin^2\theta\,d\theta$

$$\oint_{\partial R} P\,dx + Q\,dy = \int_0^{2\pi} 512\cos^2\theta\sin^2\theta\,d\theta = 128\int_0^{2\pi} \sin^2(2\theta)\,d\theta = 128\int_0^{2\pi} \frac{1-\cos(4\theta)}{2}\,d\theta$$

$$= 64\left[\theta - \frac{\sin(4\theta)}{4} \right]_0^{2\pi} = 128\pi$$

The answers are the same!