Name________________________ Sec__________

MATH 251 Honors Exam 1 Spring 2010

Sections 200 Solutions P. Yasskin

1-11	/55	14	/20
12	/10	15	/11
13	/10	Total	/106

Multiple Choice: (5 points each. No part credit.)

1. The points $A = (2,-3.4)$ and $B = (4,1,0)$ are the endpoints of the diameter of a sphere. What is the radius of the sphere?

a. 2

b. 3 Correct Choice

c. 4

d. 5

e. 6

The diameter is $d = d(A,B) = \sqrt{(4-2)^2 + (1--3)^2 + (0-4)^2} = \sqrt{4+16+16} = 6.$ The radius is $r = 3.$

2. Find a vector perpendicular to the plane containing the points

$$P = (2,1,4), \quad Q-(-1,3,2) \quad \text{and} \quad R = (3,1,2)$$

a. $(2,-1,2)$

b. $(-4,8,-2)$

c. $(2,4,1)$ Correct Choice

d. $(2,-2,1)$

e. $(-4,2,-4)$

$\overrightarrow{PQ} = Q - P = (-3,2,-2) \qquad \overrightarrow{PR} = R - P = (1,0,-2)$

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 2 & -2 \\ 1 & 0 & -2 \end{vmatrix} = \hat{\imath}(-4-0) - \hat{\jmath}(6+2) + \hat{k}(0-2) = (-4,-8,-2) \quad \text{or any multiple.}$$

3. Find the angle between the normals to the planes $3x + 2y - 4z = 3$ and $2x - y + z = 2.$

a. $0°$

b. $30°$

c. $45°$

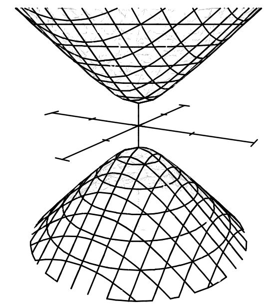
d. $60°$

e. $90°$ Correct Choice

The normals are $\vec{N}_1 = (3,2,-4)$ and $\vec{N}_2 = (2,-1,1).$

Since $\vec{N}_1 \cdot \vec{N}_2 = 6 - 2 - 4 = 0,$ the vectors are perpendicular.

4. The plot at the right is the graph of which equation?



a. $x^2+y^2-z^2 = 1$

b. $x^2+y^2-z^2 = 0$

c. $x^2+y^2-z^2 = -1$ Correct Choice

d. $x^2+y^2-z = 1$

e. $x^2+y^2-z = -1$

(c) is $x^2+y^2+1 = z^2$ So $z \ge 1$ or $z \le -1$.

5. For the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$, which of the following is FALSE?

a. $\vec{v} = \left(-e^{-t}, \sqrt{2}, e^{t}\right)$ velocity

b. $\vec{a} = (e^{-t}, 0, e^{t})$ acceleration

c. $\dfrac{ds}{dt} = e^{-t}+e^{t}$ speed

d. $a_T = e^{-t}-e^{t}$ tangential acceleration Correct Choice

e. $L = e^2-e^{-2}$ arc length between A and B

$\dfrac{ds}{dt} = |\vec{v}| = \sqrt{e^{-2t}+2+e^{2t}} = e^{-t}+e^{t}$ $\qquad a_T = \dfrac{d^2s}{dt^2} = \dfrac{d|\vec{v}|}{dt} = -e^{-t}+e^{t}$ (d is FALSE.)

$L = \int_{(1,0,1)}^{\left(e^{-2},2\sqrt{2},e^{2}\right)} ds = \int_0^2 |\vec{v}|\,dt = \int_0^2 (e^{-t}+e^{t})\,dt = [-e^{-t}+e^{t}]_0^2 = -e^{-2}+e^{2}-(-1+1) = e^2-e^{-2}$

6. Compute $\int_A^B \vec{F}\cdot d\vec{s}$ with $\vec{F} = (-z, y, x)$ along the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$.

a. $e^4-e^{-4}-1$

b. $e^2-e^{-2}-1$

c. $e^4-e^{-4}-2$

d. $e^2-e^{-2}-2$

e. 8 Correct Choice

$\vec{F} = (-z, y, x) \qquad \vec{F}(\vec{r}(t)) = \left(-e^{t}, \sqrt{2}\,t, e^{-t}\right) \qquad d\vec{s} = \vec{v}\,dt = \left(-e^{-t}, \sqrt{2}, e^{t}\right)$

$\int_A^B \vec{F}\cdot d\vec{s} = \int_0^2 \vec{F}(\vec{r}(t))\cdot\vec{v}\,dt = \int_0^2 (1+2t+1)\,dt = \int_0^2 (2+2t)\,dt = [2t+t^2]_0^2 = 8$

7. If $f(x,y,z) = y^2z^2 + x^2\sin(yz)$, which of the following is $\dfrac{\partial^3 f}{\partial z\partial y\partial x}$?

a. $-2xyz\sin(yz)$

b. $2x\cos(yz) - 2xyz\sin(yz)$ Correct Choice

c. $4yz - 2xyz\sin(yz)$

d. $2yz + 2x\cos(yz) - 2xyz\sin(yz)$

e. $4yz + 2x\cos(yz) - 2xyz\sin(yz)$

$$\frac{\partial f}{\partial x} = 2x\sin(yz) \qquad \frac{\partial^2 f}{\partial y\partial x} = 2xz\cos(yz) \qquad \frac{\partial^3 f}{\partial z\partial y\partial x} = 2x\cos(yz) - 2xyz\sin(yz)$$

8. Find the plane tangent to the graph of the function $z = x^2y^3$ at the point $(x,y) = (3,2)$.
What is the z-intercept?

a. -288 Correct Choice

b. -144

c. -72

d. 72

e. 144

$f(x,y) = x^2y^3 \quad f_x(x,y) = 2xy^3 \quad f_y(x,y) = 3x^2y^2$

$f(3,2) = 72 \qquad f_x(3,2) = 48 \qquad f_y(3,2) = 108$

$z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 72 + 48(x-3) + 108(y-2) = 48x + 108y - 288$

So the z-intercept is $c = -288$.

9. Find the plane tangent to the graph of the equation $xy + xz + yz = 11$ at the point $(x,y,z) = (3,2,1)$.
What is the z-intercept?

a. $-\dfrac{11}{5}$

b. $\dfrac{11}{5}$

c. $\dfrac{22}{5}$ Correct Choice

d. 11

e. 22

Let $f = xy + xz + yz$. Then $\vec{\nabla} f = (y+z, x+z, x+y)$.

Then the normal at $P = (3,2,1)$ is $\vec{N} = \vec{\nabla} f\big|_P = (3,4,5)$.

The equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $3x + 4y + 5z = 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 22$

Solve for $z = -\dfrac{3}{5}x - \dfrac{4}{5}y + \dfrac{22}{5}$. So the z-intercept is $c = \dfrac{22}{5}$.

10. The pressure in a certain ideal gas is given by $P = \dfrac{T}{100V}$ where the temperature is currently $T = 300°K$ and increasing at $2°K/\text{min}$ and the volume is currently $V = 4\ \text{m}^3$ and increasing at $\dfrac{1}{3}\ \text{m}^3/\text{min}$. Is the pressure increasing or decreasing and at what rate?

a. decreasing at $\dfrac{23}{400}$ atm/min Correct Choice

b. decreasing at $\dfrac{27}{400}$ atm/min

c. increasing at $\dfrac{23}{400}$ atm/min

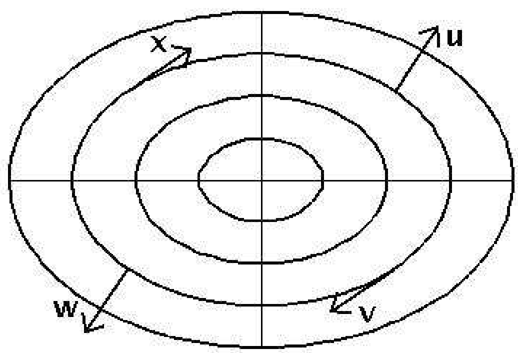
d. increasing at $\dfrac{25}{400}$ atm/min

e. increasing at $\dfrac{27}{400}$ atm/min

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{1}{100V}\frac{dT}{dt} - \frac{T}{100V^2}\frac{dV}{dt}$$

$$= \frac{1}{100 \cdot 4}2 - \frac{300}{100 \cdot 4^2}\frac{1}{3} = -\frac{23}{400} = -0.0575$$

11. The graph at the right shows the contour plot of a function $f(x,y)$ as well as several vectors. Which vectors could not be the gradient of f?



a. $\vec{u}$ and $\vec{v}$

b. $\vec{u}$ and $\vec{w}$

c. $\vec{v}$ and $\vec{w}$

d. $\vec{v}$ and $\vec{x}$ Correct Choice

e. $\vec{w}$ and $\vec{x}$

$\vec{v}$ and $\vec{x}$ are not perpendicular to the level curves.

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (10 points) A cardboard box is 5 inches long, 4 inches wide and 3 inches high.

The cardboard is 0.05 inches thick.

Use differentials to estimate the volume of cardboard used to make the box.

$V = LWH \qquad dL = dW = dH = 2 \cdot 0.05 = 0.1$

$$\Delta V \approx dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial W} dW + \frac{\partial V}{\partial H} dH = WH\,dL + LH\,dW + LW\,dH$$

$$= 4 \cdot 3 \cdot 0.1 + 5 \cdot 3 \cdot 0.1 + 5 \cdot 4 \cdot 0.1 = 4.7$$

13. (10 points) A wire has the shape of the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$. (See problem 5.) Find its mass if its linear density is given by $\rho = z - x$.

$\rho = z - x \qquad \rho(\vec{r}(t)) = e^{t} - e^{-t}$

$$M = \int_A^B \rho\, ds = \int_0^2 \rho(\vec{r}(t))|\vec{v}|\, dt = \int_0^2 (e^{t} - e^{-t})(e^{-t} + e^{t})\, dt = \int_0^2 (e^{2t} - e^{-2t})\, dt = \left[\frac{e^{2t}}{2} - \frac{e^{-2t}}{-2} \right]_0^2$$

$$= \left(\frac{e^4}{2} + \frac{e^{-4}}{2} \right) - \left(\frac{1}{2} + \frac{1}{2} \right) = \frac{e^4}{2} + \frac{e^{-4}}{2} - 1$$

14. (20 points) Duke Skywater is chasing the Dark Invader through a Dark Matter field. Duke is currently at the point $P = (3,2,1)$ and the dark matter density is $\rho = xy + xz + yz$.

a. What is the time rate of change of the dark matter density as seen by Duke if his velocity is $\vec{v} = (1,2,3)$?

$$\vec{\nabla}\rho = (y+z, x+z, x+y) \qquad \vec{\nabla}\rho\Big|_P = (3,4,5) \qquad \frac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26$$

b. In what **unit** vector direction should Duke travel to increase the dark matter density as fast as possible?

The direction of maximum increase is $\vec{\nabla}\rho\Big|_P = (3,4,5)$ and $\left|\vec{\nabla}\rho\right| = \sqrt{9+16+25} = 5\sqrt{2}$.

So the unit vector is $\dfrac{\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \left(\dfrac{3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right)$.

c. What is the maximum rate of increase of the dark matter density in any unit vector direction?

The maximum rate of increase is $\left|\vec{\nabla}\rho\right| = 5\sqrt{2}$.

15. (11 points) For each of the following limits, say whether or not the limit exists.

If it exists, give its value and prove it.

If it does not exist, give a counter example.

a. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^4+y^2}$

$$\lim_{y=mx,\ x\to 0} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2mx}{x^4+m^2x^2} = \lim_{x\to 0} \frac{mx}{x^2+m^2} = \frac{0}{m^2} = 0$$

$$\lim_{y=mx^2,\ x\to 0} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0} \frac{x^2mx^2}{x^4+m^2x^4} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2}$$

Since they are not equal when $m \neq 0$, the limit does not exist.

b. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+x^2y^4}{x^2+y^4}$

For $(x,y) \neq (0,0)$, $\dfrac{x^4+x^2y^4}{x^2+y^4} = \dfrac{x^2({}^2+y^4)}{x^2+y^4} = x^2$

So $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4+x^2y^4}{x^2+y^4} = \lim\limits_{(x,y)\to(0,0)} x^2 = 0$ and the limit exists.

c. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2}$

In polar coordinates,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+y^2} = \lim_{r\to 0,\ \theta \text{ arbitrary}} \frac{r^2\cos^2\theta\, r\sin\theta}{r^2} = \lim_{r\to 0,\ \theta \text{ arbitrary}} r\cos^2\theta\sin\theta$$

Since $r \to 0$ and $\cos^2\theta\sin\theta$ is bounded,

$\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y}{x^2+y^2} = 0$ and the limit exists.