1. The points $A = (2, -3, 4)$ and $B = (4, 1, 0)$ are the endpoints of the diameter of a sphere. What is the radius of the sphere?
   a. 2
   b. 3 Correct Choice
   c. 4
   d. 5
   e. 6

   The diameter is $d = d(A, B) = \sqrt{(4 - 2)^2 + (1 - 3)^2 + (0 - 4)^2} = \sqrt{4 + 16 + 16} = 6$. The radius is $r = 3$.

2. Find a vector perpendicular to the plane containing the points
   $P = (2, 1, 4), \quad Q = (-1, 3, 2) \quad$ and $\quad R = (3, 1, 2)$
   a. $(2, -1, 2)$
   b. $(-4, 8, -2)$
   c. $(2, 4, 1)$ Correct Choice
   d. $(2, -2, 1)$
   e. $(-4, 2, -4)$

   $\vec{PQ} = Q - P = (-3, 2, -2)$ \quad $\vec{PR} = R - P = (1, 0, -2)$

   $\vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -3 & 2 & -2 \\ 1 & 0 & -2 \end{vmatrix} = i(-4 - 0) - j(6 + 2) + k(0 - 2) = (-4, -8, -2)$ or any multiple.

3. Find the angle between the normals to the planes $3x + 2y - 4z = 3$ and $2x - y + z = 2$.
   a. $0^\circ$
   b. $30^\circ$
   c. $45^\circ$
   d. $60^\circ$
   e. $90^\circ$ Correct Choice

   The normals are $\vec{N}_1 = (3, 2, -4)$ and $\vec{N}_2 = (2, -1, 1)$.

   Since $\vec{N}_1 \cdot \vec{N}_2 = 6 - 2 - 4 = 0$, the vectors are perpendicular.
4. The plot at the right is the graph of which equation?
   
   a. $x^2 + y^2 - z^2 = 1$
   b. $x^2 + y^2 - z^2 = 0$
   c. $x^2 + y^2 - z^2 = -1$  Correct Choice
   d. $x^2 + y^2 - z = 1$
   e. $x^2 + y^2 - z = -1$

   (c) is $x^2 + y^2 + 1 = z^2$  So $z \geq 1$ or $z \leq -1$.

5. For the curve $\vec{r}(t) = \left( e^{-t}, \sqrt{2} t, e^t \right)$ between $A = (1, 0, 1)$ and $B = (e^{-2}, 2\sqrt{2}, e^2)$, which of the following is FALSE?
   
   a. $\vec{v} = (e^{-t}, \sqrt{2}, e^t)$  velocity
   b. $\vec{a} = (e^{-t}, 0, e^t)$  acceleration
   c. $\frac{ds}{dt} = e^{-t} + e^t$  speed
   d. $a_T = e^{-t} - e^t$  tangential acceleration  Correct Choice
   e. $L = e^2 - e^{-2}$  arc length between $A$ and $B$

   $\frac{ds}{dt} = |\vec{v}| = \sqrt{e^{-2t} + 2 + e^{2t}} = e^{-t} + e^t$  \hspace{1cm} $a_T = \frac{d^2s}{dt^2} = \frac{d[|\vec{v}|]}{dt} = -e^{-t} + e^t$  (d is FALSE.)

   $L = \int_{(1,0,1)}^{(e^{-2},2\sqrt{2},e^2)} ds = \int_0^2 |\vec{v}| dt = \int_0^2 (e^{-t} + e^t) dt = [-e^{-t} + e^t]_0^2 = -e^{-2} + e^2 - (-1 + 1) = e^2 - e^{-2}$

6. Compute $\int_A^B \vec{F} \cdot d\vec{s}$ with $\vec{F} = (-z, y, x)$ along the curve $\vec{r}(t) = \left( e^{-t}, \sqrt{2} t, e^t \right)$ between $A = (1, 0, 1)$ and $B = (e^{-2}, 2\sqrt{2}, e^2)$.
   
   a. $e^4 - e^{-4} - 1$
   b. $e^2 - e^{-2} - 1$
   c. $e^4 - e^{-4} - 2$
   d. $e^2 - e^{-2} - 2$
   e. 8  Correct Choice

   $\vec{F} = (-z, y, x) \hspace{1cm} \vec{F}(\vec{r}(t)) = \left( -e^{-t}, \sqrt{2} t, e^t \right) \hspace{1cm} d\vec{s} = \vec{v} dt = \left( -e^{-t}, \sqrt{2}, e^t \right)$

   $\int_A^B \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{v} dt = \int_0^2 (1 + 2t + 1) dt = \int_0^2 (2 + 2t) dt = [2t + t^2]_0^2 = 8$
7. If \( f(x, y, z) = y^2z^2 + x^2\sin(yz) \), which of the following is \( \frac{\partial^3 f}{\partial z \partial y \partial x} \)?

a. \(-2xyz\sin(yz)\)

b. \(2x\cos(yz) - 2xyz\sin(yz)\)  Correct Choice

c. \(4yz - 2xyz\sin(yz)\)

d. \(2yz + 2x\cos(yz) - 2xyz\sin(yz)\)

e. \(4yz + 2x\cos(yz) - 2xyz\sin(yz)\)

\[
\frac{\partial f}{\partial x} = 2x\sin(yz) \quad \frac{\partial^2 f}{\partial y \partial x} = 2xz\cos(yz) \quad \frac{\partial^3 f}{\partial z \partial y \partial x} = 2x\cos(yz) - 2xyz\sin(yz)
\]

8. Find the plane tangent to the graph of the function \( z = x^2y^3 \) at the point \((x, y) = (3, 2)\). What is the \(z\)-intercept?

a. \(-288\)  Correct Choice

b. \(-144\)

c. \(-72\)

d. \(72\)

e. \(144\)

\[
f(x, y) = x^2y^3 \quad f_x(x, y) = 2xy^3 \quad f_y(x, y) = 3x^2y^2
\]

\[
f(3, 2) = 72 \quad f_x(3, 2) = 48 \quad f_y(3, 2) = 108
\]

\[
z = f(3, 2) + f_x(3, 2)(x - 3) + f_y(3, 2)(y - 2) = 72 + 48(x - 3) + 108(y - 2) = 48x + 108y - 288
\]

So the \(z\)-intercept is \(c = -288\).

9. Find the plane tangent to the graph of the equation \( xy + xz + yz = 11 \) at the point \((x, y, z) = (3, 2, 1)\). What is the \(z\)-intercept?

a. \(\frac{-11}{5}\)

b. \(\frac{11}{5}\)

c. \(\frac{22}{5}\)  Correct Choice

d. \(11\)

e. \(22\)

Let \( f = xy + xz + yz \). Then \( \vec{V}f = (y + z, x + z, x + y) \).

Then the normal at \( P = (3, 2, 1) \) is \( \vec{N} = \left. \vec{V}f \right|_P = (3, 4, 5) \).

The equation of the plane is \( \vec{N} \cdot X = \vec{N} \cdot P \) or \( 3x + 4y + 5z = 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 22 \)

Solve for \( z = -\frac{3}{5}x - \frac{4}{5}y + \frac{22}{5} \). So the \(z\)-intercept is \(c = \frac{22}{5}\).
10. The pressure in a certain ideal gas is given by \( P = \frac{T}{100V} \) where the temperature is currently \( T = 300°K \) and increasing at \( 2°K/min \) and the volume is currently \( V = 4 \text{ m}^3 \) and increasing at \( \frac{1}{3} \text{ m}^3/\text{min} \).

Is the pressure increasing or decreasing and at what rate?

- a. decreasing at \( \frac{23}{400} \) atm/min Correct Choice
- b. decreasing at \( \frac{27}{400} \) atm/min
- c. increasing at \( \frac{23}{400} \) atm/min
- d. increasing at \( \frac{25}{400} \) atm/min
- e. increasing at \( \frac{27}{400} \) atm/min

\[
\frac{dP}{dt} = \frac{\partial P}{\partial T} \frac{dT}{dt} + \frac{\partial P}{\partial V} \frac{dV}{dt} = \frac{1}{100V} \frac{dT}{dt} - \frac{T}{100V^2} \frac{dV}{dt} \\
= \frac{1}{100 \cdot 4^2} - \frac{300}{100 \cdot 4^2} \frac{1}{3} = -\frac{23}{400} = -0.0575
\]

11. The graph at the right shows the contour plot of a function \( f(x,y) \) as well as several vectors. Which vectors could not be the gradient of \( f \)?

- a. \( \vec{u} \) and \( \vec{v} \)
- b. \( \vec{u} \) and \( \vec{w} \)
- c. \( \vec{v} \) and \( \vec{w} \)
- d. \( \vec{v} \) and \( \vec{x} \) Correct Choice
- e. \( \vec{w} \) and \( \vec{x} \)

\( \vec{v} \) and \( \vec{x} \) are not perpendicular to the level curves.
12. (10 points) A cardboard box is 5 inches long, 4 inches wide and 3 inches high. Use differentials to estimate the volume of cardboard used to make the box.

\[ V = LWH \quad \quad dL = dW = dH = 2 \cdot 0.05 = 0.1 \]

\[ \Delta V \approx dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial W} dW + \frac{\partial V}{\partial H} dH = WHdL + LHdW + LWdH \]

\[ = 4 \cdot 3 \cdot 0.1 + 5 \cdot 3 \cdot 0.1 + 5 \cdot 4 \cdot 0.1 = 4.7 \]

13. (10 points) A wire has the shape of the curve \( \vec{r}(t) = (e^{-t}, \sqrt{2}t, e^t) \) between \( A = (1,0,1) \) and \( B = (e^{-2}, 2\sqrt{2}, e^2) \). (See problem 5.) Find its mass if its linear density is given by \( \rho = z - x \).

\[ \rho = z - x \quad \quad \rho(\vec{r}(t)) = e^t - e^{-t} \]

\[ M = \int_A^B \rho(\vec{r}(t)) |\vec{v}| dt = \int_0^2 (e^t - e^{-t})(e^t + e^t) dt = \int_0^2 (e^{2t} - e^{-2t}) dt = \left[ \frac{e^{2t}}{2} - \frac{e^{-2t}}{-2} \right]_0^2 \]

\[ = \left( \frac{e^4}{2} + \frac{e^{-4}}{2} \right) - \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{e^4}{2} + \frac{e^{-4}}{2} - 1 \]

14. (20 points) Duke Skywater is chasing the Dark Invader through a Dark Matter field. Duke is currently at the point \( P = (3,2,1) \) and the dark matter density is \( \rho = xy + xz + yz \).

a. What is the time rate of change of the dark matter density as seen by Duke if his velocity is \( \vec{v} = (1,2,3) \)?

\[ \vec{v} \rho = (y + z, x + z, x + y) \quad \quad \vec{v} \rho \bigg|_P = (3,4,5) \quad \quad \frac{d \rho}{dt} = \vec{v} \cdot \nabla \rho = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26 \]

b. In what unit vector direction should Duke travel to increase the dark matter density as fast as possible?

The direction of maximum increase is \( \vec{\nabla} \rho \bigg|_P = (3,4,5) \) and \( |\vec{\nabla} \rho| = \sqrt{9 + 16 + 25} = 5\sqrt{2} \).

So the unit vector is \( \frac{\vec{\nabla} \rho}{|\vec{\nabla} \rho|} = \left( \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \).

c. What is the maximum rate of increase of the dark matter density in any unit vector direction?

The maximum rate of increase is \( |\vec{\nabla} \rho| = 5\sqrt{2} \).