Name__________________________ Sec__________

MATH 251 Exam 1 Spring 2010

Sections 511 Solutions P. Yasskin

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|---|---|---|---|
| 1-11 | /66 | 14 | /20 |
| 12 | /10 | | |
| 13 | /10 | Total | /106 |

Multiple Choice: (6 points each. No part credit.)

**1**. The points $A = (2,-3.4)$ and $B = (4,1,0)$ are the endpoints of the diameter of a sphere. What is the radius of the sphere?

**a**. 2

**b**. 3 Correct Choice

**c**. 4

**d**. 5

**e**. 6

The diameter is $d = d(A,B) = \sqrt{(4-2)^2 + (1--3)^2 + (0-4)^2} = \sqrt{4+16+16} = 6$. The radius is $r = 3$.

**2**. Find a vector perpendicular to the plane containing the points

$$P = (2,1,4), \quad Q-(-1,3,2) \quad \text{and} \quad R = (3,1,2)$$

**a**. $(2,-1,2)$

**b**. $(-4,8,-2)$

**c**. $(2,4,1)$ Correct Choice

**d**. $(2,-2,1)$

**e**. $(-4,2,-4)$

$\overrightarrow{PQ} = Q - P = (-3,2,-2) \qquad \overrightarrow{PR} = R - P = (1,0,-2)$

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3 & 2 & -2 \\ 1 & 0 & -2 \end{vmatrix} = \hat{\imath}(-4-0) - \hat{\jmath}(6+2) + \hat{k}(0-2) = (-4,-8,-2) \quad \text{or any multiple.}$$

**3**. Find the angle between the normals to the planes $3x + 2y - 4z = 3$ and $2x - y + z = 2$.

**a**. $0°$

**b**. $30°$

**c**. $45°$

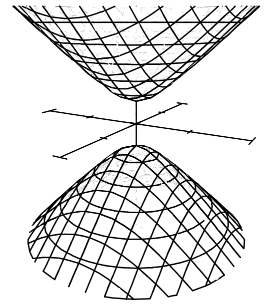
**d**. $60°$

**e**. $90°$ Correct Choice

The normals are $\vec{N}_1 = (3,2,-4)$ and $\vec{N}_2 = (2,-1,1)$.

Since $\vec{N}_1 \cdot \vec{N}_2 = 6 - 2 - 4 = 0$, the vectors are perpendicular.

**4.** The plot at the right is the graph of which equation?



**a.** $x^2 + y^2 - z^2 = 1$

**b.** $x^2 + y^2 - z^2 = 0$

**c.** $x^2 + y^2 - z^2 = -1$ Correct Choice

**d.** $x^2 + y^2 - z = 1$

**e.** $x^2 + y^2 - z = -1$

(c) is $x^2 + y^2 + 1 = z^2$ So $z \ge 1$ or $z \le -1$.

**5.** For the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$, which of the following is FALSE?

**a.** $\vec{v} = \left(-e^{-t}, \sqrt{2}, e^{t}\right)$ velocity

**b.** $\vec{a} = (e^{-t}, 0, e^{t})$ acceleration

**c.** $\dfrac{ds}{dt} = e^{-t} + e^{t}$ speed

**d.** $a_T = e^{-t} - e^{t}$ tangential acceleration Correct Choice

**e.** $L = e^2 - e^{-2}$ arc length between $A$ and $B$

$\dfrac{ds}{dt} = |\vec{v}| = \sqrt{e^{-2t} + 2 + e^{2t}} = e^{-t} + e^{t}$ $\qquad a_T = \dfrac{d^2s}{dt^2} = \dfrac{d|\vec{v}|}{dt} = -e^{-t} + e^{t}$ (d is FALSE.)

$$L = \int_{(1,0,1)}^{\left(e^{-2},2\sqrt{2},e^{2}\right)} ds = \int_0^2 |\vec{v}|\,dt = \int_0^2 (e^{-t} + e^{t})\,dt = [-e^{-t} + e^{t}]_0^2 = -e^{-2} + e^{2} - (-1+1) = e^2 - e^{-2}$$

**6.** Compute $\displaystyle\int_A^B \vec{F} \cdot d\vec{s}$ with $\vec{F} = (-z, y, x)$ along the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$.

**a.** $e^4 - e^{-4} - 1$

**b.** $e^2 - e^{-2} - 1$

**c.** $e^4 - e^{-4} - 2$

**d.** $e^2 - e^{-2} - 2$

**e.** $8$ Correct Choice

$\vec{F} = (-z, y, x) \qquad \vec{F}(\vec{r}(t)) = \left(-e^{t}, \sqrt{2}\,t, e^{-t}\right) \qquad d\vec{s} = \vec{v}\,dt = \left(-e^{-t}, \sqrt{2}, e^{t}\right)$

$$\int_A^B \vec{F} \cdot d\vec{s} = \int_0^2 \vec{F}(\vec{r}(t)) \cdot \vec{v}\,dt = \int_0^2 (1 + 2t + 1)\,dt = \int_0^2 (2 + 2t)\,dt = [2t + t^2]_0^2 = 8$$

**7**. If $f(x,y,z) = y^2z^2 + x^2\sin(yz)$, which of the following is $\dfrac{\partial^3 f}{\partial z\partial y\partial x}$?

**a**. $-2xyz\sin(yz)$

**b**. $2x\cos(yz) - 2xyz\sin(yz)$ Correct Choice

**c**. $4yz - 2xyz\sin(yz)$

**d**. $2yz + 2x\cos(yz) - 2xyz\sin(yz)$

**e**. $4yz + 2x\cos(yz) - 2xyz\sin(yz)$

$\dfrac{\partial f}{\partial x} = 2x\sin(yz)$ $\quad \dfrac{\partial^2 f}{\partial y\partial x} = 2xz\cos(yz)$ $\quad \dfrac{\partial^3 f}{\partial z\partial y\partial x} = 2x\cos(yz) - 2xyz\sin(yz)$

**8**. Find the plane tangent to the graph of the function $z = x^2y^3$ at the point $(x,y) = (3,2)$.
What is the $z$-intercept?

**a**. $-288$ Correct Choice

**b**. $-144$

**c**. $-72$

**d**. $72$

**e**. $144$

$f(x,y) = x^2y^3$ $\quad f_x(x,y) = 2xy^3$ $\quad f_y(x,y) = 3x^2y^2$

$f(3,2) = 72$ $\quad f_x(3,2) = 48$ $\quad f_y(3,2) = 108$

$z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 72 + 48(x-3) + 108(y-2) = 48x + 108y - 288$

So the $z$-intercept is $c = -288$.

**9**. Find the plane tangent to the graph of the equation $xy + xz + yz = 11$ at the point $(x,y,z) = (3,2,1)$.
What is the $z$-intercept?

**a**. $-\dfrac{11}{5}$

**b**. $\dfrac{11}{5}$

**c**. $\dfrac{22}{5}$ Correct Choice

**d**. $11$

**e**. $22$

Let $f = xy + xz + yz$. Then $\vec{\nabla} f = (y+z, x+z, x+y)$.

Then the normal at $P = (3,2,1)$ is $\vec{N} = \vec{\nabla} f\big|_P = (3,4,5)$.

The equation of the plane is $\vec{N} \cdot X = \vec{N} \cdot P$ or $3x + 4y + 5z = 3 \cdot 3 + 4 \cdot 2 + 5 \cdot 1 = 22$

Solve for $z = -\dfrac{3}{5}x - \dfrac{4}{5}y + \dfrac{22}{5}$. So the $z$-intercept is $c = \dfrac{22}{5}$.

**10**. The pressure in a certain ideal gas is given by $P = \frac{T}{100V}$ where the temperature is currently $T = 300°K$ and increasing at $2°K/\text{min}$ and the volume is currently $V = 4\text{ m}^3$ and increasing at $\frac{1}{3}\text{ m}^3/\text{min}$. Is the pressure increasing or decreasing and at what rate?

**a**. decreasing at $\frac{23}{400}$ atm/min Correct Choice

**b**. decreasing at $\frac{27}{400}$ atm/min

**c**. increasing at $\frac{23}{400}$ atm/min

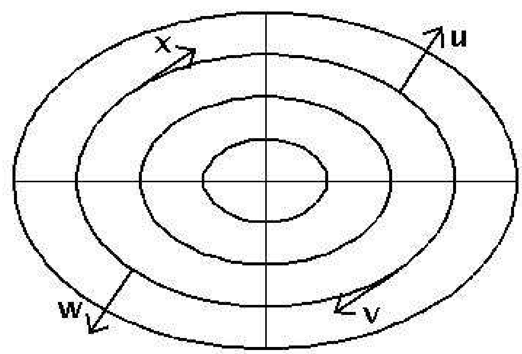
**d**. increasing at $\frac{25}{400}$ atm/min

**e**. increasing at $\frac{27}{400}$ atm/min

$$\frac{dP}{dt} = \frac{\partial P}{\partial T}\frac{dT}{dt} + \frac{\partial P}{\partial V}\frac{dV}{dt} = \frac{1}{100V}\frac{dT}{dt} - \frac{T}{100V^2}\frac{dV}{dt}$$

$$= \frac{1}{100 \cdot 4}2 - \frac{300}{100 \cdot 4^2}\frac{1}{3} = -\frac{23}{400} = -0.0575$$

**11**. The graph at the right shows the contour plot of a function $f(x,y)$ as well as several vectors. Which vectors could not be the gradient of $f$?




**a**. $\vec{u}$ and $\vec{v}$

**b**. $\vec{u}$ and $\vec{w}$

**c**. $\vec{v}$ and $\vec{w}$

**d**. $\vec{v}$ and $\vec{x}$ Correct Choice

**e**. $\vec{w}$ and $\vec{x}$

$\vec{v}$ and $\vec{x}$ are not perpendicular to the level curves.

Work Out: (Points indicated. Part credit possible. Show all work.)

**12**. (10 points) A cardboard box is $5$ inches long, $4$ inches wide and $3$ inches high.

The cardboard is $0.05$ inches thick.

Use differentials to estimate the volume of cardboard used to make the box.

$$V = LWH \qquad dL = dW = dH = 2 \cdot 0.05 = 0.1$$

$$\Delta V \approx dV = \frac{\partial V}{\partial L} dL + \frac{\partial V}{\partial W} dW + \frac{\partial V}{\partial H} dH = WH\,dL + LH\,dW + LW\,dH$$

$$= 4 \cdot 3 \cdot 0.1 + 5 \cdot 3 \cdot 0.1 + 5 \cdot 4 \cdot 0.1 = 4.7$$

**13**. (10 points) A wire has the shape of the curve $\vec{r}(t) = \left(e^{-t}, \sqrt{2}\,t, e^{t}\right)$ between $A = (1,0,1)$ and $B = \left(e^{-2}, 2\sqrt{2}, e^{2}\right)$. (See problem 5.) Find its mass if its linear density is given by $\rho = z - x$.

$$\rho = z - x \qquad \rho(\vec{r}(t)) = e^{t} - e^{-t}$$

$$M = \int_A^B \rho\, ds = \int_0^2 \rho(\vec{r}(t))|\vec{v}|\, dt = \int_0^2 (e^{t} - e^{-t})(e^{-t} + e^{t})\, dt = \int_0^2 (e^{2t} - e^{-2t})\, dt = \left[ \frac{e^{2t}}{2} - \frac{e^{-2t}}{-2} \right]_0^2$$

$$= \left( \frac{e^4}{2} + \frac{e^{-4}}{2} \right) - \left( \frac{1}{2} + \frac{1}{2} \right) = \frac{e^4}{2} + \frac{e^{-4}}{2} - 1$$

**14**. (20 points) Duke Skywater is chasing the Dark Invader through a Dark Matter field. Duke is currently at the point $P = (3,2,1)$ and the dark matter density is $\rho = xy + xz + yz$.

**a**. What is the time rate of change of the dark matter density as seen by Duke if his velocity is $\vec{v} = (1,2,3)$?

$$\vec{\nabla}\rho = (y+z, x+z, x+y) \qquad \vec{\nabla}\rho\Big|_P = (3,4,5) \qquad \frac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = 1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 = 26$$

**b**. In what **unit** vector direction should Duke travel to increase the dark matter density as fast as possible?

The direction of maximum increase is $\vec{\nabla}\rho\Big|_P = (3,4,5)$ and $\left|\vec{\nabla}\rho\right| = \sqrt{9+16+25} = 5\sqrt{2}$.

So the unit vector is $\dfrac{\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \left( \dfrac{3}{5\sqrt{2}}, \dfrac{4}{5\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right)$.

**c**. What is the maximum rate of increase of the dark matter density in any unit vector direction?

The maximum rate of increase is $\left|\vec{\nabla}\rho\right| = 5\sqrt{2}$.