Name_______________________ Sec__________

1-10	/60	12	/15
11	/10	13	/20
		Total	/105

MATH 251 Exam 2 Spring 2010

Sections 511 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. The point $(2,1)$ is a critical point of the function $f(x,y) = x^3 - 3x^2 + xy^2 - 4xy + 3x + 4y$.
Use the Second Derivative Test to classify this critical point.

a. Local Maximum

b. Local Minimum Correct Choice

c. Inflection Point

d. Saddle Point

e. Test Fails

$f_x(x,y) = 3x^2 - 6x + y^2 - 4y + 3$ $\quad f_x(2,1) = 0$

$f_y(x,y) = 2xy - 4x + 4$ $\quad f_y(2,1) = 0$

$f_{xx}(x,y) = 6x - 6$ $\quad f_{xx}(2,1) = 6 > 0$

$f_{yy}(x,y) = 2x$ $\quad f_{yy}(2,1) = 4$

$f_{xy}(x,y) = 2y - 4$ $\quad f_{xy}(2,1) = -2$

$D = f_{xx}f_{yy} - f_{xy}^2$ $\quad D(2,1) = 6 \bullet 4 - (-2)^2 = 20 > 0$ Local Minimum

2. Compute $\int_0^1 \int_0^x \int_0^y xy\,dz\,dy\,dx$

a. $\frac{1}{12}$

b. $\frac{1}{15}$ Correct Choice

c. $\frac{1}{20}$

d. $\frac{2}{15}$

e. $\frac{3}{20}$

$$\int_0^1 \int_0^x \int_0^y xy\,dz\,dy\,dx = \int_0^1 \int_0^x \left[xyz\right]_{z=0}^{y} dy\,dx = \int_0^1 \int_0^x xy^2\,dy\,dx = \int_0^1 \left[\frac{xy^3}{3}\right]_{y=0}^{x} dx = \int_0^1 \frac{x^4}{3}\,dx = \left[\frac{x^5}{15}\right]_0^1 = \frac{1}{15}$$

3. Find the mass of the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,2,0)$ and $(0,0,4)$, if the density is $\rho = x$.

a. $\frac{1}{3}$ Correct Choice

b. $\frac{4}{3}$

c. $\frac{8}{3}$

d. 3

e. 4

3 of the 4 planes are the coordinate planes. The 4th plane is $z = 4-4x-2y$ which intersects the xy-plane at $z = 4-4x-2y = 0$ or $y = 2-2x$. The mass is

$$M = \iiint \rho\, dV = \int_0^1 \int_0^{2-2x} \int_0^{4-4x-2y} x\, dz\, dy\, dx = \int_0^1 \int_0^{2-2x} x(4-4x-2y)\, dy\, dx = \int_0^1 x\Big[4y-4xy-y^2\Big]_{y=0}^{2-2x} dx$$

$$= \int_0^1 x\Big(4(2-2x)-4x(2-2x)-(2-2x)^2\Big)\, dx = \int_0^1 (4x^3-8x^2+4x)\, dx = \left[x^4 - \frac{8x^3}{3} + 2x^2\right]_0^1$$

$$= 1 - \frac{8}{3} + 2 = \frac{1}{3}$$

4. Find the volume of the solid below $z = 4-x^2-y^2$ above the xy-plane.

a. π

b. 2π

c. 4π

d. 8π Correct Choice

e. 16π

$$V = \iiint 1\, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} 1\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 (4-r^2) r\, dr = 2\pi\left[\frac{4r^2}{2} - \frac{r^4}{4}\right]_0^2 = 2\pi(8-4) = 8\pi$$

5. Find the average of the function $f = z^2$ within the solid below $z = 4-x^2-y^2$ above the xy-plane.

a. $\frac{4}{5}$

b. $\frac{8\pi}{5}$

c. $\frac{8}{3}$ Correct Choice

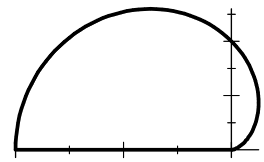
d. $\frac{16}{3}$

e. $\frac{64\pi}{3}$

$$\iiint f\, dV = \int_0^{2\pi} \int_0^2 \int_0^{4-r^2} z^2\, r\, dz\, dr\, d\theta = 2\pi \int_0^2 \left[\frac{z^3}{3}\right]_0^{4-r^2} r\, dr = 2\pi \int_0^2 \frac{(4-r^2)^3}{3} r\, dr = 2\pi\left[\frac{(4-r^2)^4}{3(-2)4}\right]_0^2$$

$$= \frac{\pi}{12}(4^4) = \frac{64\pi}{3} \qquad f_{\text{ave}} = \frac{\iiint f\, dV}{V} = \frac{64\pi}{3}\,\frac{1}{8\pi} = \frac{8}{3}$$

6. Find the volume below $z = y$ above the region between the x-axis and the upper half of the cardioid $r = 1 - \cos\theta$.



a. $\frac{1}{12}$

b. $\frac{1}{6}$

c. $\frac{2}{3}$

d. $\frac{4}{3}$ Correct Choice

e. $\frac{8}{3}$

$$V = \iint y\,dA = \int_0^\pi \int_0^{1-\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^\pi \left[\frac{r^3}{3}\right]_{r=0}^{1-\cos\theta} \sin\theta\,d\theta = \int_0^\pi \frac{(1-\cos\theta)^3}{3}\sin\theta\,d\theta$$
$$= \left[\frac{(1-\cos\theta)^4}{12}\right]_0^\pi = \frac{2^4}{12} = \frac{4}{3}$$

7. Compute $\iiint e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dV$ over the hemisphere $x^2+y^2+z^2 \le 9$ with $z \ge 0$.

a. $\frac{\pi}{3}(e^{27}-1)$

b. $\frac{\pi}{3}e^{27}$

c. $\frac{2\pi}{3}(e^{27}-1)$ Correct Choice

d. $\frac{2\pi}{3}e^{27}$

e. $\frac{4\pi}{3}(e^{27}-1)$

$$\iiint e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 e^{\rho^3}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{e^{\rho^3}}{3}\right]_0^3\left[-\cos\varphi\right]_0^{\pi/2} = \frac{2\pi}{3}(e^{27}-1)$$

8. Compute $\oint \vec{F}\cdot d\vec{s}$ for the vector field $\vec{F} = (-yz^2, xz^2, z^3)$ around the circle $\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 8)$.

a. -256π

b. -128π

c. 128π

d. 256π

e. 512π Correct Choice

$\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$ $\quad \vec{F}(\vec{r}(\theta)) = (-128\sin\theta, 128\cos\theta, 512)$

$$\oint \vec{F}\cdot d\vec{s} = \int_0^{2\pi}\vec{F}\cdot\vec{v}\,d\theta = \int_0^{2\pi}(256\sin^2\theta + 256\cos^2\theta)\,d\theta = \int_0^{2\pi} 256\,d\theta = 512\pi$$

9. Find the area of the piece of the surface $z = xy$ for (x,y) in the semicircle $x^2 + y^2 \le 9$ for $y \ge 0$.

HINT: Parametrize the surface as $\vec{R}(u,v) = (u,v,???)$.

a. $\frac{\pi}{3}(10^{3/2} - 1)$ Correct Choice

b. $\frac{2\pi}{3}(10^{3/2} - 1)$

c. 9π

d. 18π

e. 36π

$$\vec{R}(u,v) = (u,v,uv)$$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (1, & 0, & v) \\ \vec{e}_v = & (0, & 1, & u) \end{array} \qquad \vec{N} = \hat{\imath}(0 - v) - \hat{\jmath}(u - 0) + \hat{k}(1) = (-v, -u, 1) \qquad \left|\vec{N}\right| = \sqrt{v^2 + u^2 + 1}$$

$$A = \iint 1\,dS = \int\int\left|\vec{N}\right|\,du\,dv = \int\int\sqrt{v^2 + u^2 + 1}\,du\,dv$$ switch to polar:

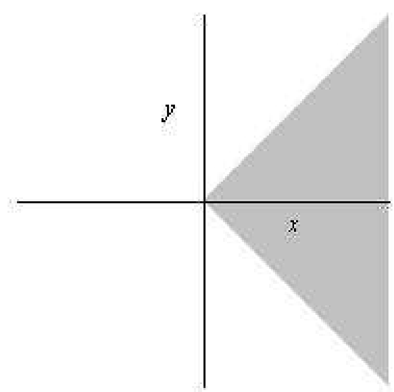
$$A = \int_0^{\pi}\int_0^3 \sqrt{r^2 + 1}\,r\,dr\,d\theta = \pi\left[\frac{(r^2+1)^{3/2}}{3}\right]_0^3 = \frac{\pi}{3}(10^{3/2} - 1)$$

10. The hyperbolic coordinate system given by

$$x = \frac{r}{2}(e^s + e^{-s}) \qquad y = \frac{r}{2}(e^s - e^{-s})$$

covers the quarter plane shown.

Find its Jacobian factor.



a. $r(e^s + e^{-s})$

b. $r\left|e^s - e^{-s}\right|$

c. $\frac{r}{2}(e^{2s} + e^{-2s})$

d. $\frac{r}{2}|e^{2s} - e^{-2s}|$

e. r Correct Choice

$$J = \left|\frac{\partial(x,y)}{\partial(r,s)}\right| = \left\|\begin{array}{cc} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \end{array}\right\| = \left\|\begin{array}{cc} \frac{1}{2}(e^s + e^{-s}) & \frac{1}{2}(e^s - e^{-s}) \\ \frac{r}{2}(e^s - e^{-s}) & \frac{r}{2}(e^s + e^{-s}) \end{array}\right\| = \left|\frac{r}{4}(e^s + e^{-s})^2 - \frac{r}{4}(e^s - e^{-s})^2\right| = r$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (10 points) Find the dimensions and volume of the largest rectangular solid box which sits on the xy-plane and has its upper vertices on the paraboloid $z = 4 - 4x^2 - y^2$.

METHOD 1: Eliminate a Variable

$V = LWH \qquad L = 2x \qquad W = 2y \qquad H = z = 4 - 4x^2 - y^2$

Maximize $V = 4xy(4 - 4x^2 - y^2) = 16xy - 16x^3y - 4xy^3$

$V_x = 16y - 48x^2y - 4y^3 = 4y(4 - 12x^2 - y^2) = 0$

$V_y = 16x - 16x^3 - 12xy^2 = 4x(4 - 4x^2 - 3y^2) = 0$

To have non-zero volume we need $x \neq 0$ and $y \neq 0$. So

$$\begin{array}{llll} 12x^2 + y^2 = 4 & \text{3•Eq1–Eq2:} & 32x^2 = 8 & x = \frac{1}{2} \\ 4x^2 + 3y^2 = 4 & \text{3•Eq2–Eq1:} & 8y^2 = 8 & y = 1 \end{array}$$

$L = 1 \qquad W = 2 \qquad H = 4 - 4\left(\frac{1}{2}\right)^2 - (1)^2 = 2 \qquad V = 4$

METHOD 2: Lagrange Multipliers

$V = LWH \qquad L = 2x \qquad W = 2y \qquad H = z$

Maximize $V = 4xyz$ subject to $g = z + 4x^2 + y^2 = 4$

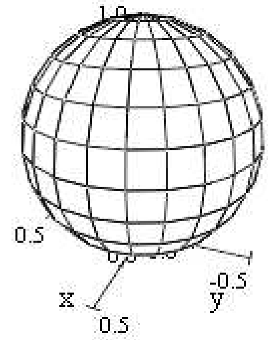
$\vec{\nabla} V = (4yz, 4xz, 4xy) \qquad \vec{\nabla} g = (8x, 2y, 1) \qquad \vec{\nabla} V = \lambda \vec{\nabla} g$

$$\begin{array}{l} 4yz = \lambda 8x \\ 4xz = \lambda 2y \\ 4xy = \lambda \end{array} \Rightarrow \begin{array}{l} 4yz = 32x^2 y \\ 4xz = 8xy^2 \end{array} \Rightarrow \begin{array}{l} z = 8x^2 \\ z = 2y^2 \end{array} \qquad g = z + 4x^2 + y^2 = z + \frac{z}{2} + \frac{z}{2} = 2z = 4$$

$z = 2 \qquad 8x^2 = 2 \qquad x = \frac{1}{2} \qquad 2y^2 = 2 \qquad y = 1$

$L = 1 \qquad W = 2 \qquad H = 2 \qquad V = 4$

12. (15 points) Find the mass and center of mass of the solid sphere of radius $\frac{1}{2}$ sitting on the xy-plane, given in spherical coordinates by $\rho = \cos\varphi$, if the mass density is $\delta = z$.



$$M = \iiint \delta\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\varphi} \rho\cos\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_{\rho=0}^{\cos\varphi}\cos\varphi\sin\varphi\, d\varphi$$

$$= \frac{\pi}{2}\int_0^{\pi/2}\cos^5\varphi\sin\varphi\, d\varphi = \frac{\pi}{2}\left[-\frac{\cos^6\varphi}{6}\right]_{\varphi=0}^{\pi/2} = \frac{\pi}{2}\left[0+\frac{1}{6}\right] = \frac{\pi}{12}$$

$\bar{x} = \bar{y} = 0$ by symmetry

$$M_{xy} = \iiint z\,\delta\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{\cos\varphi} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\int_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_{\rho=0}^{\cos\varphi}\cos^2\varphi\sin\varphi\, d\varphi$$

$$= \frac{2\pi}{5}\int_0^{\pi/2}\cos^7\varphi\sin\varphi\, d\varphi = \frac{2\pi}{5}\left[-\frac{\cos^8\varphi}{8}\right]_{\varphi=0}^{\pi/2} = \frac{2\pi}{5}\left[0+\frac{1}{8}\right] = \frac{\pi}{20}$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{\pi}{20}\frac{12}{\pi} = \frac{3}{5}$$

13. (20 points) Compute $\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S}$ for the vector field $\vec{F} = (-yz^2, xz^2, z^3)$ over the paraboloid $z = 2x^2+2y^2$ for $z \le 8$ oriented down and out.
Follow these steps. Be sure to check the orientation.

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz^2 & xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0-2xz) - \hat{\jmath}(0--2yz) + \hat{k}(z^2--z^2) = (-2xz, -2yz, 2z^2)$$

Parametrize the surface as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r^2)$

$$\begin{matrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_r = & (\cos\theta, & \sin\theta, & 4r) \\ \vec{e}_\theta = & (-r\sin\theta, & r\cos\theta, & 0) \end{matrix}$$

$$\vec{N} = \hat{\imath}(0-4r^2\cos\theta) - \hat{\jmath}(0--4r^2\sin\theta) + \hat{k}(r\cos^2\theta--r\sin^2\theta) = (-4r^2\cos\theta, -4r^2\sin\theta, r)$$

Reverse $\vec{N} = (4r^2\cos\theta, 4r^2\sin\theta, -r)$

$$\left(\vec{\nabla}\times\vec{F}\right)\left(\vec{R}(r,\theta)\right) = (-2xz, -2yz, 2z^2) = \left(-2(r\cos\theta)(2r^2), -2(r\sin\theta)(2r^2), 2(2r^2)^2\right)$$
$$= (-4r^3\cos\theta, -4r^3\sin\theta, 8r^4)$$

$$\left(\vec{\nabla}\times\vec{F}\right)\cdot\vec{N} = -16r^5\cos^2\theta - 16r^5\sin^2\theta - 8r^5 = -24r^5$$

$$\iint_P \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \iint_P \vec{\nabla}\times\vec{F}\cdot\vec{N}\, dr\, d\theta = -\int_0^{2\pi}\int_0^2 24r^5\, dr\, d\theta = -2\pi[4r^6]_0^2 = -512\pi$$