Name______________________ Sec________

1-12	/60	14	/15
13	/15	15	/15
		Total	/105

MATH 251H Exam 2 Spring 2011

Section 200 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^2 \int_0^y xy\,dx\,dy$.

a. 1

b. 2 Correct Choice

c. 3

d. 4

e. y^2

$$\int_0^2 \int_0^y xy\,dx\,dy = \int_0^2 \frac{x^2 y}{2}\Big|_{x=0}^{y} dy = \frac{1}{2}\int_0^2 y^3\,dy = \frac{1}{2}\frac{y^4}{4}\Big|_{y=0}^{2} = 2$$

2. Find the area of one loop of the rose $r = \sin(3\theta)$.

a. $\frac{\pi}{12}$ Correct Choice

b. $\frac{\pi}{6}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{3}$

e. $\frac{\pi}{2}$

$\sin(3\theta) = 0$ at $\theta = 0$ and $3\theta = \pi$ or $\theta = \pi/3$

$$A = \int_0^{\pi/3}\int_0^{\sin(3\theta)} r\,dr\,d\theta = \int_0^{\pi/3}\left[\frac{r^2}{2}\right]_{r=0}^{\sin(3\theta)} d\theta = \frac{1}{2}\int_0^{\pi/3}\sin^2(3\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3}\frac{1-\cos(6\theta)}{2}\,d\theta = \frac{1}{4}\left[\theta - \frac{\sin(6\theta)}{6}\right]_{\theta=0}^{\pi/3} = \frac{\pi}{12}$$

3. Compute $\iiint x^2 + y^2\,dV$ over the region between the cones $z = \sqrt{x^2+y^2}$ and $z = 4 - \sqrt{x^2+y^2}$.

a. $\frac{8\pi}{3}$

b. $\frac{16\pi}{3}$

c. $\frac{32\pi}{3}$

d. $\frac{16\pi}{5}$

e. $\frac{32\pi}{5}$ Correct Choice

In cylindrical coordinates, the cones are $z = r$ and $z = 4 - r$ which intersect at $r = 2$.

$$\int_0^{2\pi}\int_0^2\int_r^{4-r} r^2\,r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r^3 z\right]_{z=r}^{4-r} dr = 2\pi\int_0^2 r^3(4-2r)\,dr = 2\pi\left[r^4 - 2\frac{r^5}{5}\right]_{r=0}^{2} = 2\pi\left(16 - \frac{64}{5}\right) = \frac{32\pi}{5}$$

4. Find the mass of the hemisphere $x^2+y^2+z^2 \le 4$ with $y \ge 0$ if the density is $\delta = y$.

a. $\frac{\pi}{2}$

b. π

c. 2π

d. 4π Correct Choice

e. 8

$$M = \iiint \delta\,dV = \int_0^\pi \int_0^\pi \int_0^2 \rho\sin\varphi\sin\theta\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^2 \rho^3\,d\rho \int_0^\pi \sin\theta\,d\theta \int_0^\pi \sin^2\varphi\,d\varphi$$

$$= \left[\frac{\rho^4}{4}\right]_0^2 \left[-\cos\theta\right]_0^\pi \left[\frac{1}{2}\left(\varphi - \frac{\sin 2\varphi}{2}\right)\right]_0^\pi = 4(2)\left(\frac{\pi}{2}\right) = 4\pi$$

5. Find the center of mass of the hemisphere $x^2+y^2+z^2 \le 4$ with $y \ge 0$ if the density is $\delta = y$.

a. $\left(0, \frac{64\pi}{15}, 0\right)$

b. $\left(0, \frac{16}{15}, 0\right)$ Correct Choice

c. $\left(0, \frac{\pi^2}{12}, 0\right)$

d. $\left(0, \frac{15}{16}, 0\right)$

e. $\left(0, \frac{12}{\pi^2}, 0\right)$

$$M_{xz} = \iiint y\delta\,dV = \int_0^\pi \int_0^\pi \int_0^2 \rho^2\sin^2\varphi\sin^2\theta\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \int_0^2 \rho^4\,d\rho \int_0^\pi \sin^2\theta\,d\theta \int_0^\pi \sin^3\varphi\,d\varphi$$

$$= \left[\frac{\rho^5}{5}\right]_0^2 \left[\frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^\pi \int_0^\pi (1-\cos^2\varphi)\sin\varphi\,d\varphi = \frac{2^5}{5}\left(\frac{\pi}{2}\right)\left[-\cos\varphi + \frac{\cos^3\varphi}{3}\right]_0^\pi$$

$$= \frac{16\pi}{5}\left(1 - \frac{1}{3} + 1 - \frac{1}{3}\right) = \frac{64\pi}{15}$$

$\bar{y} = \frac{M_{xz}}{M} = \frac{64\pi}{15}\frac{1}{4\pi} = \frac{16}{15}$ $\bar{x} = \bar{z} = 0$ by symmetry.

6. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (-16x^2y, 9xy^2)$ counterclockwise around the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$.

HINTS: The ellipse may be parametrized by $\vec{r}(\theta) = (3\cos\theta, 4\sin\theta)$.

Since $\sin(2\theta) = 2\sin\theta\cos\theta$, we have $4\sin^2\theta\cos^2\theta = \sin^2(2\theta)$.

a. -864π

b. -288π

c. 144π

d. 288π

e. 864π Correct Choice

$\vec{F}(\vec{r}(\theta)) = (-16 \cdot 9\cos^2\theta \cdot 4\sin\theta, 9 \cdot 3\cos\theta \cdot 16\sin^2\theta)$ $\vec{v} = (-3\sin\theta, 4\cos\theta)$

$$\vec{F} \cdot \vec{v} = 64 \cdot 27\cos^2\theta\sin^2\theta + 27 \cdot 64\cos^2\theta\sin^2\theta = 27 \cdot 64 \cdot 2\cos^2\theta\sin^2\theta = 27 \cdot 32\sin^2(2\theta)$$

$$= 27 \cdot 32\frac{1-\cos(4\theta)}{2} = 27 \cdot 16(1-\cos(4\theta))$$

$$\oint \vec{F} \cdot d\vec{s} = \oint \vec{F} \cdot \vec{v}\,d\theta = \int_0^{2\pi} 27 \cdot 16(1-\cos(4\theta))\,d\theta = 27 \cdot 16\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = 864\pi$$

7. The point $\left(\frac{\pi}{3}, \frac{\pi}{6}\right)$ is a critical point of the function $f(x,y) = \sin(x)\cos(y) - \frac{\sqrt{3}}{4}x + \frac{\sqrt{3}}{4}y$
Use the Second Derivative Test to classify this critical point.

a. Local Maximum Correct Choice

b. Local Minimum

c. Inflection Point

d. Saddle Point

e. Test Fails

$$f_x = \cos(x)\cos(y) - \frac{\sqrt{3}}{4} \qquad f_x\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{4} = 0$$

$$f_y = -\sin(x)\sin(y) + \frac{\sqrt{3}}{4} \qquad f_y\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) + \frac{\sqrt{3}}{4} = 0$$

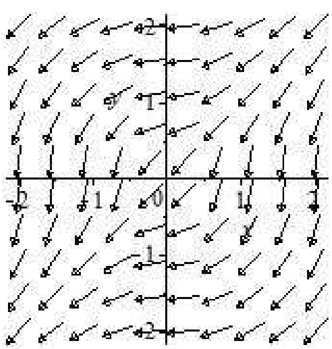
$$f_{xx} = -\sin(x)\cos(y) \qquad f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = -\frac{3}{4} < 0$$

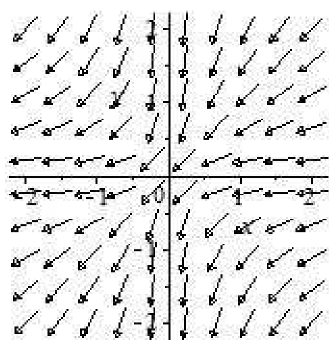
$$f_{yy} = -\sin(x)\cos(y) \qquad f_{yy}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) = -\frac{3}{4}$$

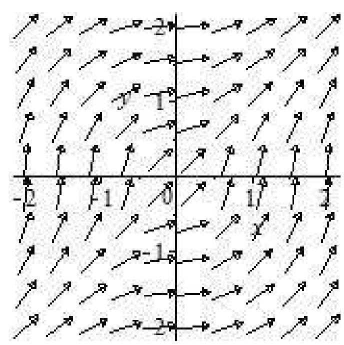
$$f_{xy} = -\cos(x)\sin(y) \qquad f_{xy}\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = -\cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right) = -\frac{1}{4}$$

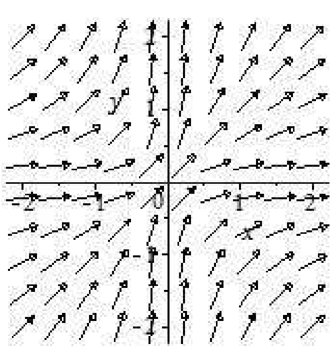
$$D = f_{xx}f_{yy} - f_{xy}^2 \qquad D\left(\frac{\pi}{3}, \frac{\pi}{6}\right) = \left(-\frac{3}{4}\right)\left(-\frac{3}{4}\right) - \left(-\frac{1}{4}\right)^2 = \frac{1}{2} > 0 \quad \text{Local Maximum}$$

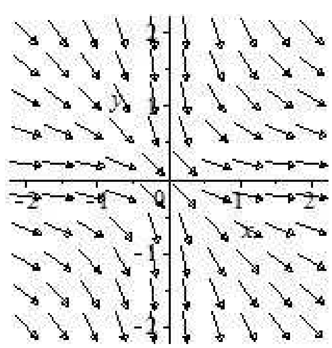
8. Which of the following is the plot of the vector field $\vec{F}(x,y) = \frac{1}{\sqrt{x^2+y^2}}\left(|x|,\ |y| \right)$?

a. 

b. 

c. 

d. 

e. 

(d) Correct Choice

All arrows must be up and right. So (c) or (d). On the x-axis, $y = 0$ and so the arrows are horizontal there. So (d).

9. In $\mathbb{R}^4$, consider the parameric 2-surface $(x,y,z,w) = \vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2r\cos\theta, 2r\sin\theta)$ for $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Compute $\iint\limits_{\vec{R}} (xz\,dy\,dz - y^2\,dz\,dw)$.

a. 32π
b. 16π
c. -16π
d. -32π Correct Choice
e. -64π

$$dy\,dz = \frac{\partial(y,z)}{\partial(r,\theta)}\,dr\,d\theta = \begin{vmatrix} \sin\theta & 2\cos\theta \\ r\cos\theta & -2r\sin\theta \end{vmatrix} dr\,d\theta = (-2r\sin^2\theta - 2r\cos^2\theta)\,dr\,d\theta = -2r\,dr\,d\theta$$

$$dz\,dw = \frac{\partial(z,w)}{\partial(r,\theta)}\,dr\,d\theta = \begin{vmatrix} 2\cos\theta & 2\sin\theta \\ -2r\sin\theta & 2r\cos\theta \end{vmatrix} dr\,d\theta = (4r\cos^2\theta + 4r\sin^2\theta)\,dr\,d\theta = 4r\,dr\,d\theta$$

$$xz = 2r^2\cos^2\theta \qquad -y^2 = -r^2\sin^2\theta$$

$$\iint_{\vec{R}} (xz\,dy\,dz + y^2\,dz\,dw) = \iint (-2r^2\cos^2\theta 2r\,dr\,d\theta - r^2\sin^2\theta 4r\,dr\,d\theta) = -4\int_0^{2\pi}\int_0^2 (r^3)\,dr\,d\theta = -8\pi\left[\frac{r^4}{4}\right]_0^2 = -32\pi$$

10. Find the equation of the plane tangent to the parameric surface $\vec{R}(u,t) = \left(ue^t, ue^{-t}, \sqrt{2}\,u\right)$ at the point $P = \vec{R}(2,0)$ where $u = 2$ and $t = 0$.
Hint: Evaluate the normal $\vec{N}$ at $u = 2$ and $t = 0$.

a. $x + y - \sqrt{2}\,z = -4\sqrt{2}$
b. $x + y - \sqrt{2}\,z = 0$ Correct Choice
c. $x + y - \sqrt{2}\,z = 16\sqrt{2}$
d. $\sqrt{2}\,x - \sqrt{2}\,y + 2z = -4\sqrt{2}$
e. $\sqrt{2}\,x - \sqrt{2}\,y + 2z = 0$

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (e^t & e^{-t} & \sqrt{2}) \\ \vec{e}_t = & (ue^t & -ue^{-t} & 0) \end{array} \quad \vec{N} = \vec{e}_u \times \vec{e}_t = \hat{\imath}\left(\sqrt{2}\,ue^{-t}\right) - \hat{\jmath}\left(-\sqrt{2}\,ue^t\right) + \hat{k}(-u-u) = \left(\sqrt{2}\,ue^{-t}, \sqrt{2}\,ue^t, -2u\right)$$

$$P = \vec{R}(2,0) = \left(2,2,2\sqrt{2}\right) \qquad \vec{N} = \left(\sqrt{2}\,ue^{-t}, \sqrt{2}\,ue^t, -2u\right) = \left(2\sqrt{2}, 2\sqrt{2}, -4\right)$$

$$\vec{N}\cdot X = \vec{N}\cdot P \qquad 2\sqrt{2}\,x + 2\sqrt{2}\,y - 4z = 2\sqrt{2}\,(2) + 2\sqrt{2}\,(2) - 4\left(2\sqrt{2}\right) = 0$$

11. If $\vec{F} = (xy\tan z, yz\cos x, xz\sin y)$, then $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} =$

- **a**. $-2z\cos y - 2x(\tan^2 z + 1)$
- **b**. $-2z\cos y + 2x(\tan^2 z + 1)$
- **c**. $2z\cos y - 2x(\tan^2 z + 1)$
- **d**. $2z\cos y + 2x(\tan^2 z + 1)$
- **e**. 0 Correct Choice

$\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ for any twice differentiable vector field.

12. Let f be the scalar potential for $\vec{F} = (2xz - 3y, 8yz - 3x, x^2 + 4y^2 + 2z)$ for which $f(0,0,0) = 0$. Then $f(1,1,1) =$

- **a**. 1
- **b**. 2
- **c**. 3 Correct Choice
- **d**. 4
- **e**. 5

$\vec{\nabla} f = \vec{F}$ or (1) $\partial_x f = 2xz - 3y$ (2) $\partial_y f = 8yz - 3x$ (3) $\partial_z f = x^2 + 4y^2 + 2z$

(1) $\Rightarrow$ $f = x^2 z - 3xy + g(y,z)$ $\Rightarrow$ (4) $\partial_y f = -3x + \partial_y g$

(2) and (4) $\Rightarrow$ $\partial_y g = 8yz$ $\Rightarrow$ $g = 4y^2 z + h(z)$ $\Rightarrow$ $f = x^2 z - 3xy + 4y^2 z + h(z)$

$\Rightarrow$ (5) $\partial_z f = x^2 + 4y^2 + \dfrac{dh(z)}{dz}$

(3) and (5) $\Rightarrow$ $\dfrac{dh(z)}{dz} = 2z$ $\Rightarrow$ $h = z^2 + C$ $\Rightarrow$ $f = x^2 z - 3xy + 4y^2 z + z^2 + C$

To have $f(0,0,0) = 0$ $\Rightarrow$ $C = 0$. So $f(1,1,1) = 1 - 3 + 4 + 1 = 3$.

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (15 points) The plane $x+2y+4z=8$ intersects the 1st octant $(x>0, y>0, z>0)$ in a triangle. Find the point on this triangle at which the function $f = xy^2z^3$ is a maximum.

Method of Eliminating a Variable:

$x = 8-2y-4z \qquad f = (8-2y-4z)y^2z^3 = 8y^2z^3 - 2y^3z^3 - 4y^2z^4$

$f_y = 16yz^3 - 6y^2z^3 - 8yz^4 = 0 \qquad \Rightarrow \qquad 2yz^3(8-3y-4z) = 0$

$f_z = 24y^2z^2 - 6y^3z^2 - 16y^2z^3 = 0 \qquad \Rightarrow \qquad 2y^2z^2(12-3y-8z) = 0$

Since $f = xy^2z^3$ is positive in the 1st octant, we know the solution cannot have $y=0$ or $z=0$.

So we solve $8-3y-4z=0$ and $12-3y-8z=0$.

Solving each for $3y$ and equating gives $3y = 8-4z = 12-8z$, or $4z=4$. So $z=1$.

Then $3y=4$. So $y=\frac{4}{3}$, and $x = 8-2\left(\frac{4}{3}\right)-4(1) = \frac{4}{3}$.

The point is $\left(\frac{4}{3},\frac{4}{3},1\right)$.

Method of Lagrange Multipliers:

$f = xy^2z^3 \qquad \vec{\nabla}f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x+2y+4z \qquad \vec{\nabla}g = (1,2,4)$

Lagrange equations: $y^2z^3 = \lambda$, $2xyz^3 = 2\lambda$, $3xy^2z^2 = 4\lambda$

Substitute the first eq into the other two: $2xyz^3 = 2y^2z^3$, $3xy^2z^2 = 4y^2z^3$.

Since $f = xy^2z^3$ is positive in the 1st octant, we know the solution cannot have $x=0$ or $y=0$ or $z=0$. So we cancel: $x=y$, $3x=4z$, So $y=x$ and $z=\frac{3}{4}x$.

Substitute into the constraint: $x+2y+4z=8 \quad x+2x+3x=8 \quad 6x=8 \quad x=\frac{4}{3} \quad y=\frac{4}{3} \quad z=1$.

The point is $\left(\frac{4}{3},\frac{4}{3},1\right)$.

14. (15 points) Compute $\iiint \vec{\nabla}\cdot\vec{F}\,dV$ for $\vec{F} = (xy^2, yz^2, zx^2)$ over the solid above the cone $z=\sqrt{x^2+y^2}$ below the sphere $x^2+y^2+z^2=4$.

$\vec{\nabla}\cdot\vec{F} = y^2+z^2+x^2 = \rho^2$ in spherical coordinates and $dV = \rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$

The cone is $\rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta} = \rho\sin\varphi$. So $\tan\varphi = 1$ or $\varphi = \pi/4$.

$$\iiint \vec{\nabla}\cdot\vec{F}\,dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^2 \rho^2\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi = \left[\frac{\rho^5}{5}\right]_0^2(2\pi)\left[-\cos\varphi\right]_0^{\pi/4} = \frac{64\pi}{5}\left(1-\frac{1}{\sqrt{2}}\right)$$

15. (15 points) Compute $\iint_E \vec{\nabla} \times \vec{F} \cdot \hat{k}\, dx\, dy$ for $\vec{F} = (-16x^2 y, 9xy^2, 0)$ over the interior of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$.

HINTS: First compute $\vec{\nabla} \times \vec{F} \cdot \hat{k}$ in rectangular coordinates.

Then compute the integral in elliptic coordinates $x = 3u\cos\theta$, $y = 4u\sin\theta$.

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -16x^2 y & 9xy^2 & 0 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(0) + \hat{k}(9y^2 - -16x^2)$$

$$\vec{\nabla} \times \vec{F} \cdot \hat{k} = 9y^2 + 16x^2 = 9(4u\sin\theta)^2 + 16(3u\cos\theta)^2 = 9 \cdot 16u^2 = 144u^2$$

$$\frac{\partial(x,y)}{\partial(u,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 3\cos\theta & 4\sin\theta \\ -3u\sin\theta & 4u\cos\theta \end{vmatrix} = 12u \qquad dx\, dy = J\, du\, d\theta = 12u\, du\, d\theta$$

$$\iint_E \vec{\nabla} \times \vec{F} \cdot \hat{k}\, dx\, dy = \int_0^{2\pi} \int_0^1 144u^2\, 12u\, du\, d\theta = 2\pi 144 \cdot 12 \left[\frac{u^4}{4} \right]_0^1 = 864\pi$$