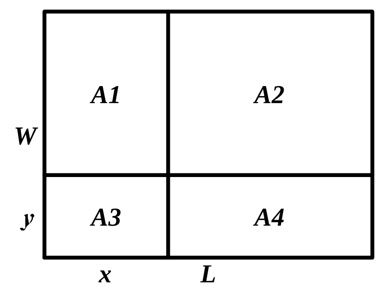
1. Minimal Rectangles and Triangles

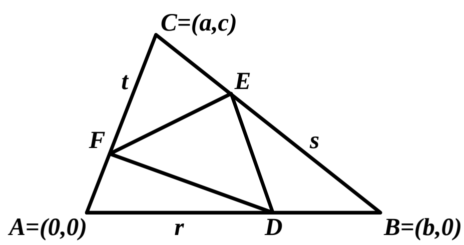
a. Consider a rectangle of length L and width W. Draw a line parallel to each side at distances x and y from one corner, as shown in the diagram:



This divides the rectangle into 4 subrectangles. Find the values of x and y which maximize and minimize the sum of the squares of the areas:

$$f = (A_1)^2 + (A_2)^2 + (A_3)^2 + (A_4)^2$$

b. Consider a triangle with vertices at $A = (0,0)$, $B = (b,0)$ and $C = (a,c)$ where a, b and c are fixed. Pick point D a fraction r of the way from A to B, point E a fraction s of the way from B to C, and point F a fraction t of the way from C to A and connect D, E and F, as shown in the diagram:



This divides the triangle into 4 subtriangles. Find the values of r, s and t which maximize and minimize the sum of the squares of the areas:

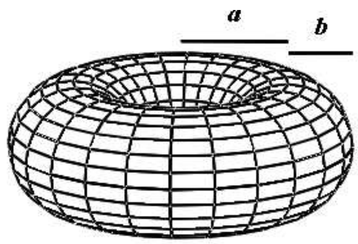
$$f = (A_1)^2 + (A_2)^2 + (A_3)^2 + (A_4)^2$$

c. In both problems, be sure to identify the configuration space and check both the interior and boundary of the configuration space for the absolute maximum and minimum.

2. Skimpy Donut

You are the mathematics consultant for a donut company which makes donuts which have a thin layer of chocolate icing covering the entire donut. One day you decide to point out that the company might cut costs on chocolate icing if they keep the volume (and hence weight) of the donut fixed but adjust the shape of the donut to minimize the surface area. Alternatively, they could advertise extra icing by maximizing the surface area. You need to write a report presenting your ideas which can be read by both the company president and the technical engineers.

A donut has the shape of a torus which is specified by giving a big radius a from the center of the hole to the center of the ring and a small radius b which is the radius of the ring, as shown in the figure.



Your job is to determine the values of a and b which extremize the surface area while keeping the volume fixed at the volume of the typical donut mentioned above. This original donut has $a = 5$ cm and $b = 3$ cm.

a. The surface of a torus satisfies the equation

$$(r - a)^2 + z^2 = b^2$$

in cylindrical coordinates where, of course, $b \le a$.

i. Compute the volume V of the torus as a function of a and b.
HINT: Integrate in cylindrical coordinates.

ii. Check that the volume of the original donut is
$V = 90\pi^2$ cm^3 ≈ 888 cm^3.

b. The surface of the torus can also be parametrized as

$$\vec{R}(\theta, \varphi) = ((a + b\cos\varphi)\cos\theta, (a + b\cos\varphi)\sin\theta, b\sin\varphi)$$

for $0 \le \theta \le 2\pi$ and $0 \le \varphi \le 2\pi$. Here, θ represents the angle around the circle of radius a and φ represents the angle around the circle of radius b.

i. Compute the surface area S of the torus as a function of a and b.
HINT: Do a surface integral in θ and φ.

ii. Check that the surface area of the original donut is
$S = 60\pi^2$ cm^2 ≈ 592 cm^2.

c. Keep the volume fixed at $V = 90\pi^2$ cm^{3}and find the values of a, b and S which minimize and maximize the surface area S. (Apply the second derivative test to any critical point in the interior and check the values at the endpoints.)

d. Write a letter to the CEO of the donut company summarizing your results (including minimum and maximum dimensions, a description of these donuts and the percent savings or extra cost). Anything you say in this report must be documented in an appendix of Maple computations for the engineers.

3. The Hypervolume of a Hypersphere

In this project, you will determine the hypervolume enclosed by a hypersphere in $\mathbb{R}^n$ whose equation is:

$$x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 + x_n{}^2 = R^2$$

a. Compute the area enclosed by the circle $x^2 + y^2 = R^2$ using a double integral in polar coordinates. Repeat using a double integral in rectangular coordinates with x as the inner integral and y as the outer integral.

Let $V_2(R)$ denote this function, where V_2 means 2-dimensional volume (area).

b. Compute the volume enclosed by the sphere $x^2 + y^2 + z^2 = R^2$ using a triple integral in spherical coordinates. Repeat using a triple integral in rectangular coordinates with x as the inner integral, y as the middle integral and z as the outer integral.

Let $V_3(R)$ denote this function, where V_3 means 3-dimensional volume.

Explain (geometrically and algebraically) why the inner 2 integrals are just $V_2\left(\sqrt{R^2 - z^2}\right)$. (Think about volume by slicing.)

c. Compute the 4-dimensional hypervolume enclosed by the hypersphere $x^2 + y^2 + z^2 + w^2 = R^2$ using a quadruple integral in rectangular coordinates with x as the inner integral, y and z as the middle integrals and w as the outer integral.

Let $V_4(R)$ denote this function, where V_4 means 4-dimensional hypervolume.

Explain (geometrically and algebraically) why the inner 3 integrals are just $V_3\left(\sqrt{R^2 - w^2}\right)$.

d. For $n = 5, 6, \cdots, 10$, find the n-dimensional hypervolume enclosed by the n-dimensional hypersphere $x_1{}^2 + x_2{}^2 + \cdots + x_{n-1}{}^2 + x_n{}^2 = R^2$.

Let $V_n(R)$ denote this function, where V_n means n-dimensional hypervolume.

HINT: If you write this volume as an n-fold integral in rectangular coordinates with x_n as the outer integral then the inner $n-1$ integrals are $V_{n-1}\left(\sqrt{R^2 - x_n{}^2}\right)$.

Explain this in terms of volume by slicing.

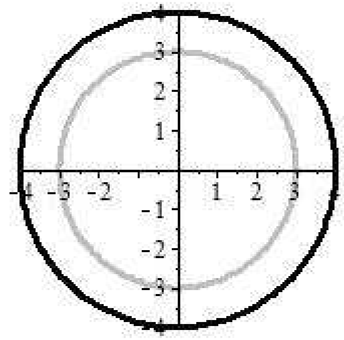
e. Looking at your results for the hypervolumes of the n-dimensional hyperspheres, deduce two general patterns for $V_n(R)$. (The formulas for n even and for n odd are different.) Explain how you got your formulas. Does your "odd" formula hold for the case $n = 1$ that is, for the length of the interval $[-R, R]$?

f. Use mathematical induction to prove your two formulas for $V_n(R)$. (Use the hint from part (d) twice.) This may be hard; so get help from your instructor.

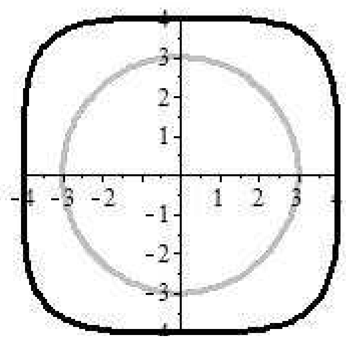
4. **Average Temperatures**

A 6 inch burner on an electric stove occupies the circle $x^2 + y^2 = 9$ with $z = 0$ and is kept at the constant temperature of $120°$C. A frying pan, a skillet and a pot of water are placed on this burner. You are to find their average temperatures (exactly if possible and approximately to 5 decimals otherwise). Their geometries and their temperatures are given below.

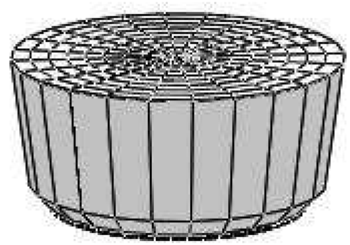
a. The 8 inch frying pan occupies the circle $x^2 + y^2 = 16$ with $z = 0$. The temperature in polar coordinates is $T = 120 - 3|r - 3|$. The frying pan and the burner are shown here:



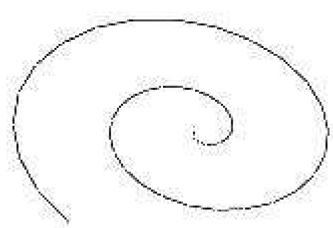
b. The 8 inch "square" skillet occupies the region inside $x^4 + y^4 = 256$. Verify this is the polar curve $r = \dfrac{4}{\sqrt[4]{\cos^4(\theta) + \sin^4(\theta)}}$ The temperature is $T = 120 - 3|r - 3|$. (You may need to split the integral for $r < 3$ and $r > 3$.



c. The water in the 8 inch pot which is 4 inches deep occupies the region given in cylindrical coordinates by $\sqrt[4]{175} - \sqrt[4]{256 - r^4} \le z \le \sqrt[4]{175}$ and has temperature $T = 120 - 3\sqrt{(r-3)^2 + z^2}$.

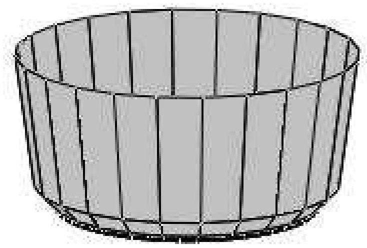


d. A temperature probe is placed in the pot of water of part (c) and measures the average temperature along a spiral curve parametrized by $\vec{r}(t) = (t\cos(t\pi), t\sin(t\pi), 2)$ for $0 \le t \le 4$. The temperature is given in cylindrical coordinates by $T = 120 - 3\sqrt{(r-3)^2 + z^2}$.



(What is the percent error between the temperature measured by the probe and the actual average temperature of thewater from part (c).)

e. The pot from part (c) occupies the surface given by $z = \sqrt[4]{175} - \sqrt[4]{256 - r^4}$ for $r \le 4$ in cylindrical coordinates and may be parametrized by $\vec{R}(r,\theta)$
$$= \left(r\cos(\theta), r\sin(\theta), \sqrt[4]{175} - \sqrt[4]{256 - r^4}\right)$$
for $r \le 4$ and $0 \le \theta \le 2\pi$. The temperature is given in cylindrical coordinates by
$T = 120 - 3\sqrt{(r-3)^2 + z^2}$.



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5. **The Volume Between a Surface and Its Tangent Plane**

In this project, you will be finding the tangent plane to a surface for which the volume between the surface and the tangent plane is a minimum.

a. Consider the surface

$$z = f(x,y) = 4x^2 + y^2 + x^2y^2$$

Verify that the surface is everywhere concave up on the square $0 \le x \le 1$ and $0 \le y \le 1$.

Note: A function fx,y is everywhere concave up on a region if $D = f_{xx}f_{yy} - {f_{xy}}^2 > 0$ and $f_{xx} > 0$ everywhere on the region. It is everywhere concave down on a region if $D = f_{xx}f_{yy} - {f_{xy}}^2 > 0$ and $f_{xx} < 0$ everywhere on the region.

b. Find its tangent plane at a general point $\left(a, b, f(a,b)\right)$.

c. Compute the volume between the surface and its general tangent plane above the square $0 \le x \le 1$ and $0 \le y \le 1$. Call this volume $V(a,b)$.

d. Find the point (a,b) for which the volume $V(a,b)$ is a minimum. Be sure to apply the second derivative test to verify that your critical point is a minimum.

e. Repeat steps (a)-(d) for two or three other functions $f(x,y)$. Use interesting functions, not just polynomials with at least one concave up and one concave down. Check the concavity.

f. What do you conjecture?

g. Prove your conjecture by repeating steps (a)-(d) for an undefined function $f(x,y)$.

h. What happens to your conjecture if you change the square base to another region R? Try some shapes other than a rectangle or a circle!

6. Gauss' Law and Ampere's Law

In this project you will calculate the electric charge inside a sphere for several electric fields using both the differential and integral versions of Gauss' Law and discuss their equivalence. Then you will calculate the electric current through a disk for several magnetic fields using both the differential and integral versions of Ampere's Law and discuss their equivalence.

Gauss' Law:

- The differential form of Gauss' Law gives the charge density as $\rho = \frac{1}{4\pi}\vec{\nabla} \cdot \vec{E}$ from which the charge inside a sphere is $Q = \iiint\limits_V \rho\, dV$ where V is the interior of the sphere.
- The integral form of Gauss' Law gives the charge inside a sphere as $Q = \frac{1}{4\pi} \iint\limits_S \vec{E} \cdot dS$ where S is the surface of the sphere.

a. Explain why Gauss' Theorem says the two charge formulas are the same.

b. Use both versions to compute the charge inside the sphere $x^2 + y^2 + z^2 = a^2$ for each of the electric fields:
$\vec{E}_n = cr^n\vec{r} = \left(c\sqrt{x^2+y^2+z^2}^{\,n}x, c\sqrt{x^2+y^2+z^2}^{\,n}y, c\sqrt{x^2+y^2+z^2}^{\,n}z\right)$ for $n = 1, 0, -1, -2, -3$.
Here, r is the length of the position vector $\vec{r} = (x, y, z)$. Use spherical coordinates to do the volume intehral and parametrize the surface of the sphere as $\vec{R}(\theta, \varphi) = (a\sin(\varphi)\cos(\theta), a\sin(\varphi)\sin(\theta), a\cos(\varphi))$.

c. For which electric field(s) is the charge density a constant? How could this constant be calculated from the charge and volume?

d. For which electric field(s) do the differential and integral forms give different answers? Why does this not violate Gauss' Theorem? In this case, the physicists regard the integral form of Gauss' Law as giving the correct answer and interpret the net charge Q as a point charge at the origin. Explain why this interpretation is reasonable by looking at the charge density at points other than the origin.

Ampere's Law:

- The differential form of Ampere's Law gives the current density as $\vec{J} = \frac{1}{4\pi}\vec{\nabla} \times \vec{B}$ from which the current through a disk is $I = \iint\limits_D \vec{J} \cdot d\vec{S}$ where D is the surface of the disk.
- The integral form of Ampere's Law gives the current through a disk as $I = \frac{1}{4\pi} \oint\limits_C \vec{B} \cdot d\vec{s}$ where C is the circle bounding the disk.

a. Explain why Stokes' Theorem says the two current formulas are the same.

b. Use both versions to compute the current through the disk $x^2 + y^2 = a^2$ for each of the electric fields:
$\vec{B}_n = 2c(r^\perp)^n\vec{r}^\perp = \left(-2c\sqrt{x^2+y^2}^{\,n}y, 2c\sqrt{x^2+y^2}^{\,n}x, 0\right)$ for $n = 1, 0, -1, -2, -3$.
Here $\vec{r}^\perp = (x, y, 0)$ is the vector from the z-axis to the point and $r^\perp$ is its length. Use polar coordinates with $z = 0$ to do the surface intehral and parametrize the circle for the line integral.

c. For which magnetic field(s) is the current density a constant? How could this constant be calculated from the current and area?

d. For which magnetic field(s) do the differential and integral forms give different answers? Why does this not violate Stokes' Theorem? In this case, the physicists regard the integral form of Ampere's Law as giving the correct answer and interpret the net current I as a current moving along the z-axis. Explain why this interpretation is reasonable by looking at the current density at points not on the z-axis.

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7. Locating an Apartment

Upon moving to a new city, you want to find an apartment which is conveniently located relative to your school, your place of work and the shopping mall. These are located at $S = (-2,-2)$ $W = (4,1)$ $M = (1,5)$ respectively. If your apartment is at $A = (x,y)$ find the location of your apartment which minimizes $f = \left|\overrightarrow{AS}\right| + \left|\overrightarrow{AW}\right| + \left|\overrightarrow{AM}\right|$. Here $\left|\overrightarrow{AS}\right|$ is the distance from your apartment to school (i.e. the length of the vector $\overrightarrow{AS}$) and similarly for $\left|\overrightarrow{AW}\right|$ and $\left|\overrightarrow{AM}\right|$. In the course of solving this problem, you should answer the following questions:

a. Compute the gradient of $\left|\overrightarrow{AS}\right|$ and express your answer in terms of the vector $\overrightarrow{AS}$. In particular, how are their directions related, how are their magnitudes related?

b. Draw a contour plot of $\left|\overrightarrow{AS}\right|$. What does it say about the direction and magnitude of the gradient $\left|\overrightarrow{AS}\right|$?

c. Find the point A which minimizes f.

d. Plot the three vectors $\overrightarrow{AS}$, $\overrightarrow{AW}$ and $\overrightarrow{AM}$ together with the triangle $\triangle SWM$.

e. Looking at your plot, give a geometric condition on the three vectors $\overrightarrow{AS}$, $\overrightarrow{AW}$ and $\overrightarrow{AM}$ which characterizes the point A which minimizes f which you feel would be true even if S, W or M were moved. Verify it for the given values.

f. What happens if the point M is moved to the right so that the angle $\angle SWM$ is greater than $135°$? Move M until you find the critical angle. Draw a series of plots or animate your plots.

g. Prove the geometric condition you found in part (e) for general position of S, W and M . It may be useful to use your results from part (a).

8. Interpretation of the Divergence and Curl

The usual formulas for the divergence and curl are derivative formulas. However there are also integral formulas for each of them which provide a more intuitive understanding of their meaning.

Divergence:

- The derivative definition of the divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$ is $\vec{\nabla} \cdot \vec{F} = \dfrac{\partial F_1}{\partial x} + \dfrac{\partial F_2}{\partial y} + \dfrac{\partial F_3}{\partial z}$.
- The integral definition of the divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$ is
$$\left(\operatorname{div}\vec{F}\right)(P) = \lim_{R\to 0}\left[\frac{3}{4\pi R^3}\iint_{S_R(P)} \vec{F} \cdot d\vec{S}\right]$$
where $\left(\operatorname{div}\vec{F}\right)(P)$ denotes the value of the divergence at a point $P = (a, b, c)$ and $S_R(P)$ is the sphere of radius R centered at P which may be parametrized by
$$\vec{R}(\theta, \varphi) = (a + R\sin(\varphi)\cos(\theta), b + R\sin(\varphi)\sin(\theta), c + R\cos(\varphi)).$$
- Interpretation: If $\vec{F}$ is the velocity field of a fluid, then the flux $\iint \vec{F} \cdot d\vec{S}$ represents the amount of fluid flowing out of the sphere per unit time. Since $V = \dfrac{4\pi R^3}{3}$ is the volume of the sphere, $\dfrac{1}{V}\iint \vec{F} \cdot d\vec{S}$ represents the amount of fluid flowing out of the sphere per unit time, per unit volume. In the limit as $R \to 0$, the $\left(\operatorname{div}\vec{F}\right)(P)$ measures amount of fluid flowing out of the point P.
- Notation: Use $\vec{\nabla} \cdot \vec{F}$ to denote the derivative definition and $\operatorname{div}\vec{F}$ to denote the integral definition.
- With this notation, Gauss' Theorem says
$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \iint_{\partial V} \vec{F} \cdot d\vec{S}$$

a. Use Gauss' Theorem to prove $\left(\operatorname{div}\vec{F}\right)(P) = \left(\vec{\nabla} \cdot \vec{F}\right)(P)$. You may assume that $\vec{\nabla} \cdot \vec{F}$ is continuous, so that its value inside a small sphere may be approximated by its value at the center of the sphere.

b. For each of the following vector fields, calculate both the derivative and integral definitions of the divergence at a point $P = (a, b, c)$.

i. $\vec{F} = (x^3, y^3, z^3)$

ii. $\vec{G} = (-yz, xz, z^2)$

iii. $\vec{u} = (xy, yz, zx)$

iv. $\vec{v} = (yz, -xz, xy)$

Curl: