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1-12	/60
13	/10
14	/10
15	/10
16	/10
Total	/100

MATH 251 Exam 1 Version A Spring 2013

Sections 506 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the line through $P = (1,2,3)$ which is perpendicular to both of the vectors $\vec{a} = \langle 3,-1,2\rangle$ and $\vec{b} = \langle 1,0,-2\rangle$.

a. $(x,y,z) = (2+t, 8+2t, 1+3t)$

b. $(x,y,z) = (2+t, -8+2t, 1+3t)$

c. $(x,y,z) = (2-t, -8-2t, 1-3t)$

d. $(x,y,z) = (1+2t, 2-8t, 3+t)$

e. $(x,y,z) = (1+2t, 2+8t, 3+t)$ Correct Choice

SOLUTION: $\vec{v} = \vec{a}\times\vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = \langle 2,8,1\rangle$ $\quad X = P + t\vec{v}$ $\quad (x,y,z) = (1+2t, 2+8t, 3+t)$

2. A triangle has vertices at $A = \langle 1,1,1\rangle$, $B = \langle 3,4,-3\rangle$ and $C = \langle 3,3,2\rangle$.
Drop a perpendicular from B to the side $\overline{AC}$.
Find the point P where the perpendicular intersects the side $\overline{AC}$.

a. $\left\langle \frac{7}{3}, \frac{7}{3}, \frac{5}{3} \right\rangle$ Correct Choice

b. $\left\langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle$

c. $\left\langle \frac{5}{3}, \frac{5}{3}, \frac{4}{3} \right\rangle$

d. $\left\langle \frac{41}{29}, \frac{47}{29}, \frac{5}{29} \right\rangle$

e. $\left\langle \frac{12}{29}, \frac{18}{29}, \frac{-24}{29} \right\rangle$

SOLUTION: $\overrightarrow{AB} = B - A = \langle 2,3,-4\rangle$ $\quad \overrightarrow{AC} = C - A = \langle 2,2,1\rangle$

$\text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AC}\right|^2}\overrightarrow{AC} = \dfrac{4+6-4}{4+4+1}\langle 2,2,1\rangle = \dfrac{2}{3}\langle 2,2,1\rangle = \left\langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle$

$P = A + \text{proj}_{\overrightarrow{AC}}\overrightarrow{AB} = \langle 1,1,1\rangle + \left\langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \right\rangle = \left\langle \frac{7}{3}, \frac{7}{3}, \frac{5}{3} \right\rangle$

3. If $\vec{u}$ points NorthEast and $\vec{v}$ points Down, then $\vec{u}\times\vec{v}$ points

- **a.** NorthEast
- **b.** NorthWest   Correct Choice
- **c.** SouthEast
- **d.** SouthWest
- **e.** Up

SOLUTION: Fingers of right hand point NorthEast with palm Down. The thumb points NorthWest.

4. Identify the quadratic surface for the equation

$$2(x-2)^2+(y-3)^2+(z-2)^2 = (x-2)^2+2(y-3)^2+(z+2)^2$$

- **a.** hyperboloid of 1 sheet
- **b.** hyperboloid of 2 sheets
- **c.** cone
- **d.** hyperbolic paraboloid   Correct Choice
- **e.** hyperbolic cylinder

SOLUTION: Subtract the right side from the left side, expand the z terms and then solve for z:

$$(x-2)^2-(y-3)^2+(z-2)^2-(z+2)^2 = 0$$

$$(x-2)^2-(y-3)^2-4z = 0 \qquad z = \frac{(x-2)^2}{4}-\frac{(y-3)^2}{4}$$

5. A girl scout is hiking up a mountain whose attitude is given by $z = h(x,y) = 10-x-x^2-y^2$. If she is currently at the point $(x,y) = (1,2)$, in what unit vector direction should she walk to go up hill as fast as possible?

- **a.** $\left(-\frac{3}{5},-\frac{4}{5}\right)$ Correct Choice
- **b.** $\left(\frac{3}{5},\frac{4}{5}\right)$
- **c.** $\left(-\frac{4}{5},-\frac{3}{5}\right)$
- **d.** $\left(\frac{4}{5},\frac{3}{5}\right)$
- **e.** $(4,3)$

SOLUTION: $\vec{\nabla}h = (-1-2x,-2y)$ $\vec{v} = \vec{\nabla}h\Big|_{(1,2)} = (-1-2,-4) = (-3,-4)$

$|\vec{v}| = \sqrt{9+16} = 5 \qquad \hat{v} = \left(-\frac{3}{5},-\frac{4}{5}\right)$

6. Find the arclength of 4 revolutions around the helix $\vec{r}(t) = (2\cos 2t, \quad 2\sin 2t, \quad 3t)$.
NOTE: Each revolution covers an angle of 2π. How much does t change?

a. 2π

b. 4π

c. 5π

d. 15π

e. 20π Correct Choice

SOLUTION: $\vec{v} = \langle -4\sin 2t, \quad 4\cos 2t, \quad 3\rangle$ $\quad |\vec{v}| = \sqrt{16\sin^2 2t + 16\cos^2 2t + 9} = 5$

We cover 1 revolution as t runs from 0 to π.

$$L = \int ds = \int |\vec{v}|\,dt = \int_0^{4\pi} 5\,dt = [5t]_0^{4\pi} = 20\pi$$

7. A wire in the shape of the helix $\vec{r}(t) = (2\cos 2t, \quad 2\sin 2t, \quad 3t)$ has linear mass density $\rho = z^2$.
Find its total mass between $t = 0$ and $t = 2\pi$.

a. $M = 24\pi^3$

b. $M = 36\pi^2$

c. $M = 120\pi^3$ Correct Choice

d. $M = 180\pi^2$

e. $M = 240\pi^2$

SOLUTION: $\rho = z^2 = 9t^2$ $\quad |\vec{v}| = 5$

$$M = \int \rho\,ds = \int z^2|\vec{v}|\,dt = \int_0^{2\pi} 9t^2 5\,dt = \left[45\frac{t^3}{3}\right]_0^{2\pi} = 15 \cdot 8\pi^3 = 120\pi^3$$

8. Find the work done to move an object along the helix $\vec{r}(t) = (2\cos 2t, \quad 2\sin 2t, \quad 3t)$
between $t = 0$ and $t = 2\pi$ by the force $\vec{F} = \langle -yz, \quad xz, \quad z\rangle$.

a. $\dfrac{33}{2}\pi$

b. 33π

c. $\dfrac{33}{2}\pi^2$

d. $33\pi^2$

e. $66\pi^2$ Correct Choice

SOLUTION: $\vec{F}(\vec{r}(t)) = \langle -6t\sin 2t, \quad 6t\cos 2t, \quad 3t\rangle$ $\quad \vec{v} = \langle -4\sin 2t, \quad 4\cos 2t, \quad 3\rangle$

$$W = \int \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v}\,dt = \int_0^{2\pi} (24t\sin^2 2t + 24t\cos^2 2t + 9t)\,dt = \int_0^{2\pi} 33t\,dt = \left[\frac{33}{2}t^2\right]_0^{2\pi} = 66\pi^2$$

9. Find the tangent line to the helix $\vec{r}(t) = (2\cos 2t, \quad 2\sin 2t, \quad 3t)$ at the point $t = \frac{\pi}{2}$.

Where does it intersect the xy-plane?

HINT : What are the position and tangent vector at $t = \frac{\pi}{2}$?

a. $(x,y) = (-1,\pi)$

b. $(x,y) = (-1,-\pi)$

c. $(x,y) = (-2,2\pi)$ Correct Choice

d. $(x,y) = (-2,-2\pi)$

e. $(x,y) = (2,\pi)$

SOLUTION: $P = \vec{r}\left(\frac{\pi}{2}\right) = \left(2\cos\pi, \quad 2\sin\pi, \quad \frac{3\pi}{2}\right) = \left(-2, 0, \frac{3\pi}{2}\right)$

$\vec{v}(t) = \langle -4\sin 2t, \quad 4\cos 2t, \quad 3\rangle \qquad \vec{v}\left(\frac{\pi}{2}\right) = \langle -4\sin\pi, \quad 4\cos\pi, \quad 3\rangle = \langle 0,-4,3\rangle$

Tangent Line: $X = P + t\vec{v} \qquad (x,y,z) = \left(-2,-4t, \frac{3\pi}{2} + 3t\right)$

The line intersects the xy-plane when $z = \frac{3\pi}{2} + 3t = 0$ or $t = -\frac{\pi}{2}$. So $(x,y) = (-2,2\pi)$

10. Find the plane tangent to the graph of $z = y\ln x$ at the point $(e,2)$. Its z-intercept is

a. $-e$

b. -2 Correct Choice

c. 0

d. 2

e. e

SOLUTION:

$f = y\ln x \qquad f(e,2) = 2 \qquad z = f(e,2) + f_x(e,2)(x-e) + f_y(e,2)(y-2)$

$f_x = \frac{y}{x} \qquad f_x(e,2) = \frac{2}{e} \qquad = 2 + \frac{2}{e}(x-e) + 1(y-2)$

$f_y = \ln x \qquad f_y(e,2) = 1$ When $x = y = 0$, we have $z = 2-2-2 = -2$.

11. Find the plane tangent to the graph of $x^2z^2 + 2zy^2 + yx^3 = 71$ at the point $(2,1,0)$. Its z-intercept is

a. 2
b. 4
c. 8
d. 16 Correct Choice
e. 32

SOLUTION: $F(x,y,z) = x^2z^2 + 2zy^2 + yx^3$ $\vec{\nabla}F = \langle 2xz^2 + 3yx^2, 4zy + x^3, 2x^2z + 2y^2 \rangle$

$\vec{N} = \vec{\nabla}F\Big|_{(2,1,0)} = \langle 12,8,2 \rangle$ $\vec{N} \cdot X = \vec{N} \cdot P$ $12x + 8y + 2z = 12 \cdot 2 + 8 \cdot 1 + 2 \cdot 0 = 32$

When $x = y = 0$, we have $z = 16.$

12. The point $(x,y) = \left(1, \frac{1}{2}\right)$ is a critical point of the function $f(x,y) = 4xy - x^3y - 4xy^3$. Use the Second Derivative Test to classify this critical point.

a. local minimum
b. local maximum Correct Choice
c. saddle point
d. TEST FAILS

SOLUTION:

$f_x = 4y - 3x^2y - 4y^3 \quad \Rightarrow \quad f_x\left(1, \frac{1}{2}\right) = 4\left(\frac{1}{2}\right) - 3\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3 = 0$ Checked

$f_y = 4x - x^3 - 12xy^2 \quad \Rightarrow \quad f_y\left(1, \frac{1}{2}\right) = 4 - 1 - 12\left(\frac{1}{2}\right)^2 = 0$ Checked

$f_{xx} = -6xy \quad \Rightarrow \quad f_{xx}\left(1, \frac{1}{2}\right) = -3$

$f_{yy} = -24xy \quad \Rightarrow \quad f_{yy}\left(1, \frac{1}{2}\right) = -12$

$f_{xy} = 4 - 3x^2 - 12y^2 \quad \Rightarrow \quad f_{xy}\left(1, \frac{1}{2}\right) = 4 - 3 - 12\left(\frac{1}{2}\right)^2 = -2$

$D = f_{xx}f_{yy} - {f_{xy}}^2 = (-3) \cdot (-12) - (-2)^2 = 32$

Since $D > 0$ and $f_{xx} < 0$ it is a local maximum.

Work Out: (10 points each. Part credit possible. Show all work.)

13. Find the point where the line $(x,y,z) = (4 + 3t, 3 - 2t, 2 + t)$ intersects the plane $x + 2y + 3z = 20$, or explain why they are parallel.

SOLUTION:

Substitute the line into the plane and solve for t:

$20 = x + 2y + 3z = (4 + 3t) + 2(3 - 2t) + 3(2 + t) = 2t + 16 = 20 \quad \Rightarrow \quad t = 2$

Substitute back into the line:

$(x,y,z) = (4 + 3(2), 3 - 2(2), 2 + (2)) = (10,-1,4)$

Check by substituting into the plane:

$x + 2y + 3z = (10) + 2(-1) + 3(4) = 20$

14. Find the line where the planes $-2x - 6y + 4z = 7$ and $3x + 9y - 6z = 5$ intersect, or explain why they are parallel.

SOLUTION:
The normal to the first plane is $\vec{N}_1 = (-2,-6,4)$.
The normal to the second plane is $\vec{N}_2 = (3,9,-6)$.
Notice that $\vec{N}_2 = -\frac{3}{2}\vec{N}_1$, so the normals are parallel.
Alternatively, compute $\vec{N}_2 \times \vec{N}_1 = \vec{0}$, so the normals are parallel.
In either case the planes are parallel and do not intersect.

15. If two adjustable resistors, with resistances R_1 and R_2, are arranged in parallel, the total resistance R is given by

$$R = \frac{R_1 R_2}{R_1 + R_2}$$

Currently, $R_1 = 3\Omega$ and $R_2 = 7\Omega$ and they are changing according to $\frac{dR_1}{dt} = -0.1 \frac{\Omega}{\sec}$ and $\frac{dR_2}{dt} = 0.2 \frac{\Omega}{\sec}$. Find R and $\frac{dR}{dt}$. Is R increasing or decreasing?

SOLUTION: $R = \frac{3 \cdot 7}{3+7} = 2.1\Omega$.

$$\frac{dR}{dt} = \frac{\partial R}{\partial R_1}\frac{dR_1}{dt} + \frac{\partial R}{\partial R_2}\frac{dR_2}{dt} = \frac{(R_1+R_2)R_2 - R_1R_2(1)}{(R_1+R_2)^2}\frac{dR_1}{dt} + \frac{(R_1+R_2)R_1 - R_1R_2(1)}{(R_1+R_2)^2}\frac{dR_2}{dt}$$

$$= \frac{(R_2)^2}{(R_1+R_2)^2}\frac{dR_1}{dt} + \frac{(R_1)^2}{(R_1+R_2)^2}\frac{dR_2}{dt} = \frac{7^2}{(3+7)^2}(-0.1) + \frac{3^2}{(3+7)^2}(.2) = \frac{-4.9+1.8}{100}$$

$= -0.031 \frac{\Omega}{\sec}$ So R is decreasing.

16. A rectangular box sits on the xy-plane with its top 4 vertices in the paraboloid $z = 8 - 2x^2 - 8y^2$. Find the dimensions and volume of the largest such box.

SOLUTION: Let the corner in the first quadrant be (x,y,z). The dimensions are $L = 2x$, $W = 2y$, $H = z$. So x, y and z are positive. So the volume is

$V = (2x)(2y)z = 4xyz = 4xy(8 - 2x^2 - 8y^2) = 32xy - 8x^3y - 32xy^3$

$V_x = 32y - 24x^2y - 32y^3 = 8y(4 - 3x^2 - 4y^2) = 0 \Rightarrow$ Since $y \neq 0$, $3x^2 + 4y^2 = 4$ (1)

$V_y = 32x - 8x^3 - 96xy^2 = 8x(4 - x^2 - 12y^2) = 0 \Rightarrow$ Since $x \neq 0$, $x^2 + 12y^2 = 4$ (2)

$3 \times (1) - (2):\quad 8x^2 = 8 \Rightarrow x = 1$

$3 \times (2) - (1):\quad 32y^2 = 8 \Rightarrow y = \frac{1}{2} \Rightarrow z = 8 - 2x^2 - 8y^2 = 8 - 2 - 2 = 4$

$L = 2x = 2.\quad W = 2y = 1,\quad H = z = 4 \qquad V = 2 \cdot 1 \cdot 4 = 8$

V is positive on the region $2x^2 + 8y^2 < 8$ with $x > 0$ and $y > 0$ and $V = 0$ on the boundary.
Since there is only one critical point, it must be a maximum.
Note: Problem 12 shows V is a local maximum.