MATH 251 Exam 1 Version B Spring 2013
Sections 506 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the line through \( P = (1, 2, 3) \) which is perpendicular to both of the vectors \( \vec{a} = \langle 3, -1, 2 \rangle \) and \( \vec{b} = \langle 1, 0, -2 \rangle \).

   a. \( (x, y, z) = (1 + 2t, 2 + 8t, 3 + t) \) Correct Choice
   b. \( (x, y, z) = (1 + 2t, 2 - 8t, 3 + t) \)
   c. \( (x, y, z) = (2 - t, -8 - 2t, 1 - 3t) \)
   d. \( (x, y, z) = (2 + t, -8 + 2t, 1 + 3t) \)
   e. \( (x, y, z) = (2 + t, 8 + 2t, 1 + 3t) \)

   SOLUTION: \( \vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 2 \\ 1 & 0 & -2 \end{vmatrix} = \langle 2, 8, 1 \rangle \) \( X = P + \vec{v} \) \( (x, y, z) = (1 + 2t, 2 + 8t, 3 + t) \)

2. A triangle has vertices at \( A = \langle 1, 1, 1 \rangle \), \( B = \langle 3, 4, -3 \rangle \) and \( C = \langle 3, 3, 2 \rangle \). Drop a perpendicular from \( B \) to the side \( \overline{AC} \). Find the point \( P \) where the perpendicular intersects the side \( \overline{AC} \).

   a. \( \langle \frac{12}{29}, \frac{18}{29}, \frac{-24}{29} \rangle \)
   b. \( \langle \frac{41}{29}, \frac{47}{29}, \frac{5}{29} \rangle \)
   c. \( \langle \frac{5}{3}, \frac{5}{3}, \frac{4}{3} \rangle \)
   d. \( \langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \rangle \)
   e. \( \langle \frac{7}{3}, \frac{7}{3}, \frac{5}{3} \rangle \) Correct Choice

   SOLUTION: \( \overrightarrow{AB} = B - A = \langle 2, 3, -4 \rangle \) \( \overrightarrow{AC} = C - A = \langle 2, 2, 1 \rangle \)
   \( \text{proj}_{\overrightarrow{AC}} \overrightarrow{AB} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AC}^2} \overrightarrow{AC} = \frac{4 + 6 - 4}{4 + 4 + 1} \langle 2, 2, 1 \rangle = \frac{2}{9} \langle 2, 2, 1 \rangle = \langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \rangle \)
   \( P = A + \text{proj}_{\overrightarrow{AC}} \overrightarrow{AB} = \langle 1, 1, 1 \rangle + \langle \frac{4}{3}, \frac{4}{3}, \frac{2}{3} \rangle = \langle \frac{7}{3}, \frac{7}{3}, \frac{5}{3} \rangle \)
3. If \( \vec{u} \) points NorthEast and \( \vec{v} \) points Down, then \( \vec{u} \times \vec{v} \) points

a. SouthWest
b. SouthEast
c. NorthWest Correct Choice
d. NorthEast
e. Up

SOLUTION: Fingers of right hand point NorthEast with palm Down. The thumb points NorthWest.

4. Identify the quadratic surface for the equation

\[ 2(x - 2)^2 + (y - 3)^2 + (z - 2)^2 = (x - 2)^2 + 2(y - 3)^2 + (z + 2)^2 \]

a. hyperboloid of 1 sheet
b. hyperboloid of 2 sheets
c. cone
d. hyperbolic paraboloid Correct Choice
e. hyperbolic cylinder

SOLUTION: Subtract the right side from the left side, expand the \( z \) terms and then solve for \( z \):

\[ (x - 2)^2 - (y - 3)^2 + (z - 2)^2 - (z + 2)^2 = 0 \]

\[ (x - 2)^2 - (y - 3)^2 - 4z = 0 \]

\[ z = \frac{(x - 2)^2}{4} - \frac{(y - 3)^2}{4} \]

5. A girl scout is hiking up a mountain whose attitude is given by \( z = h(x, y) = 10 - x - x^2 - y^2 \). If she is currently at the point \((x, y) = (1, 2)\), in what unit vector direction should she walk to go up hill as fast as possible?

a. \((4, 3)\)
b. \((\frac{4}{5}, \frac{3}{5})\)
c. \((\frac{-3}{5}, \frac{-4}{5})\) Correct Choice
d. \((\frac{-4}{5}, \frac{-3}{5})\)
e. \((\frac{3}{5}, \frac{4}{5})\)

SOLUTION: \( \vec{v}h = (-1 - 2x, -2y) \)

\[ \vec{v} = \vec{v}h \bigg|_{(1,2)} = (-1 - 2, -4) = (-3, -4) \]

\[ |\vec{v}| = \sqrt{9 + 16} = 5 \]

\[ \hat{v} = \left( -\frac{3}{5}, -\frac{4}{5} \right) \]
6. Find the arclength of 4 revolutions around the helix  \( \vec{r}(t) = (2\cos 2t, 2\sin 2t, 3t) \).

NOTE: Each revolution covers an angle of 2\( \pi \). How much does \( t \) change?

a. 20\( \pi \) Correct Choice  
b. 15\( \pi \)  
c. 5\( \pi \)  
d. 4\( \pi \)  
e. 2\( \pi \)

SOLUTION:

\[ \vec{v} = \langle -4\sin 2t, 4\cos 2t, 3 \rangle \]

\[ |\vec{v}| = \sqrt{16\sin^2 2t + 16\cos^2 2t + 9} = 5 \]

We cover 1 revolution as \( t \) runs from 0 to \( \pi \).

\[ L = \int ds = \int |\vec{v}| \, dt = \int_0^{4\pi} 5 \, dt = [5t]_0^{4\pi} = 20\pi \]

7. A wire in the shape of the helix  \( \vec{r}(t) = (2\cos 2t, 2\sin 2t, 3t) \) has linear mass density \( \rho = z^2 \). Find its total mass between \( t = 0 \) and \( t = 2\pi \).

a. \( M = 24\pi^3 \)  
b. \( M = 120\pi^3 \) Correct Choice  
c. \( M = 36\pi^2 \)  
d. \( M = 180\pi^2 \)  
e. \( M = 240\pi^2 \)

SOLUTION:

\[ \rho = z^2 = 9t^2 \]

\[ |\vec{v}| = 5 \]

\[ M = \int \rho \, ds = \int z^2 |\vec{v}| \, dt = \int_0^{2\pi} 9t^2 5 \, dt = \left[ \frac{45t^3}{3} \right]_0^{2\pi} = 15 \cdot 8\pi^3 = 120\pi^3 \]

8. Find the work done to move an object along the helix  \( \vec{r}(t) = (2\cos 2t, 2\sin 2t, 3t) \) between \( t = 0 \) and \( t = 2\pi \) by the force  \( \vec{F} = (-yz, xz, z) \).

a. \( \frac{33}{2}\pi \)  
b. 33\( \pi \)  
c. \( \frac{33}{2}\pi^2 \)  
d. 33\( \pi^2 \)  
e. 66\( \pi^2 \) Correct Choice

SOLUTION:

\[ \vec{F}(\vec{r}(t)) = \langle -6t\sin 2t, 6t\cos 2t, 3t \rangle \]

\[ |\vec{v}| = \langle -4\sin 2t, 4\cos 2t, 3 \rangle \]

\[ \vec{F} \cdot ds = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{v} \, dt = \int_0^{2\pi} (24t\sin^2 2t + 24t\cos^2 2t + 9t) \, dt = \int_0^{2\pi} 33t \, dt = \left[ \frac{33}{2} t^2 \right]_0^{2\pi} = 66\pi^2 \]
9. Find the tangent line to the helix  \( \vec{r}(t) = (2 \cos 2t, \ 2 \sin 2t, \ 3t) \) at the point \( t = \frac{\pi}{2} \).

Where does it intersect the xy-plane?

HINT: What are the position and tangent vector at \( t = \frac{\pi}{2} \)?

a. \((x,y) = (-2,-2\pi)\)

b. \((x,y) = (-2,2\pi)\) Correct Choice

c. \((x,y) = (-1,-\pi)\)

d. \((x,y) = (-1,\pi)\)

e. \((x,y) = (2,\pi)\)

SOLUTION: \( P = \vec{r}\left(\frac{\pi}{2}\right) = (2 \cos \frac{\pi}{2}, \ 2 \sin \frac{\pi}{2}, \ \frac{3\pi}{2}) = (-2, 0, \frac{3\pi}{2}) \)

\( \vec{v}(t) = (-4 \sin 2t, \ 4 \cos 2t, \ 3) \) \( \vec{v}\left(\frac{\pi}{2}\right) = (-4 \sin \pi, \ 4 \cos \pi, \ 3) = (0, -4, 3) \)

Tangent Line: \( X = P + t\vec{v} \) \( (x,y,z) = (-2, -4t, \frac{3\pi}{2} + 3t) \)

The line intersects the xy-plane when \( z = \frac{3\pi}{2} + 3t = 0 \) or \( t = -\frac{\pi}{2} \). So \((x,y) = (-2,2\pi)\)

10. Find the plane tangent to the graph of \( z = y \ln x \) at the point \((e,2)\). Its \( z \)-intercept is

a. \( e \)

b. \( 2 \)

c. \( 0 \)

d. \(-2\) Correct Choice

e. \(-e\)

SOLUTION:

\( f = y \ln x \) \( f(e,2) = 2 \) \( z = f(e,2) + f_x(e,2)(x-e) + f_y(e,2)(y-2) \)

\( f_x = \frac{y}{x} \) \( f_x(e,2) = \frac{2}{e} \) \( = 2 + \frac{2}{e}(x-e) + 1(y-2) \)

\( f_y = \ln x \) \( f_y(e,2) = 1 \) When \( x = y = 0 \), we have \( z = 2 - 2 - 2 = -2 \).
11. Find the plane tangent to the graph of \( x^2z^2 + 2zy^2 + yx^3 = 71 \) at the point \((2,1,0)\). Its \( z \)-intercept is

a. 32
b. 16 Correct Choice
c. 8
d. 4
e. 2

SOLUTION: 
\[ F(x,y,z) = x^2z^2 + 2zy^2 + yx^3 \]
\[ \vec{V}F = \langle 2xz^2 + 3yx^2, 4zy + x^3, 2x^2z + 2y^2 \rangle \]
\[ \vec{N} = \frac{\vec{V}F}{|\vec{V}F|} \bigg|_{(2,1,0)} = \langle 12, 8, 2 \rangle \]
\[ \vec{N} \cdot X = \vec{N} \cdot P \]
\[ 12x + 8y + 2z = 12 \cdot 2 + 8 \cdot 1 + 2 \cdot 0 = 32 \]

When \( x = y = 0 \), we have \( z = 16 \).

12. The point \( (x,y) = \left(1, \frac{1}{2}\right) \) is a critical point of the function \( f(x,y) = 4xy - x^3y - 4xy^3 \). Use the Second Derivative Test to classify this critical point.

a. local maximum Correct Choice
b. local minimum
c. saddle point
d. TEST FAILS

SOLUTION:
\[ f_x = 4y - 3x^2y - 4y^3 \quad \Rightarrow \quad f_x \left(1, \frac{1}{2}\right) = 4 \left( \frac{1}{2} \right) - 3 \left( \frac{1}{2} \right)^2 - 4 \left( \frac{1}{2} \right)^3 = 0 \]
Checked
\[ f_y = 4x - x^3 - 12xy^2 \quad \Rightarrow \quad f_y \left(1, \frac{1}{2}\right) = 4 - \left( \frac{1}{2} \right)^3 = 0 \]
Checked
\[ f_{xx} = -6xy \quad \Rightarrow \quad f_{xx} \left(1, \frac{1}{2}\right) = -3 \]
\[ f_{xy} = -24xy \quad \Rightarrow \quad f_{xy} \left(1, \frac{1}{2}\right) = -12 \]
\[ f_{yy} = 4 - 3x^2 - 12y^2 \quad \Rightarrow \quad f_{yy} \left(1, \frac{1}{2}\right) = 4 - 3 \left( \frac{1}{2} \right)^2 = -2 \]
\[ D = f_{xx}f_{yy} - f_{xy}^2 = (-3) \cdot (-12) - (-2)^2 = 32 \]
Since \( D > 0 \) and \( f_{xx} < 0 \) it is a local maximum.

Work Out: (10 points each. Part credit possible. Show all work.)

13. Find the line where the planes \(-2x - 6y + 4z = 7\) and \(3x + 9y - 6z = 5\) intersect, or explain why they are parallel.

SOLUTION:
The normal to the first plane is \( \vec{N}_1 = (-2,-6,4) \).
The normal to the second plane is \( \vec{N}_2 = (3,9,-6) \).
Notice that \( \vec{N}_2 = -\frac{3}{2} \vec{N}_1 \), so the normals are parallel.
Alternatively, compute \( \vec{N}_2 \times \vec{N}_1 = 0 \), so the normals are parallel.
In either case the planes are parallel and do not intersect.
14. Find the point where the line \((x, y, z) = (4 + 3t, 3 - 2t, 2 + t)\) intersects the plane \(x + 2y + 3z = 20\), or explain why they are parallel.

**SOLUTION:**
Substitute the line into the plane and solve for \(t\):
\[20 = x + 2y + 3z = (4 + 3t) + 2(3 - 2t) + 3(2 + t) = 2t + 16 = 20 \implies t = 2\]
Substitute back into the line:
\[(x, y, z) = (4 + 3(2), 3 - 2(2), 2 + (2)) = (10, -1, 4)\]
Check by substituting into the plane:
\[x + 2y + 3z = (10) + 2(-1) + 3(4) = 20\]

15. A rectangular box sits on the \(xy\)-plane with its top 4 vertices in the paraboloid \(z = 8 - 2x^2 - 8y^2\). Find the dimensions and volume of the largest such box.

**SOLUTION:** Let the corner in the first quadrant be \((x, y, z)\). The dimensions are \(L = 2x\), \(W = 2y\), \(H = z\). So \(x, y\) and \(z\) are positive. So the volume is
\[V = (2x)(2y)z = 4xyz = 4xy(8 - 2x^2 - 8y^2) = 32xy - 8x^3y - 32xy^3\]
\[V_x = 32y - 24x^2y - 32y^3 = 8y(4 - 3x^2 - 4y^2) = 0 \implies \text{Since } y \neq 0, \ 3x^2 + 4y^2 = 4 \quad (1)\]
\[V_y = 32x - 8x^3 - 96xy^2 = 8x(4 - x^2 - 12y^2) = 0 \implies \text{Since } x \neq 0, \ x^2 + 12y^2 = 4 \quad (2)\]
\[3 \times (1) - (2) : \ 8x^2 = 8 \implies x = 1\]
\[3 \times (2) - (1) : \ 32y^2 = 8 \implies y = \frac{1}{2} \implies z = 8 - 2x^2 - 8y^2 = 8 - 2 - 2 = 4\]
\[L = 2x = 2, \ W = 2y = 1, \ H = z = 4 \implies V = 2 \cdot 1 \cdot 4 = 8\]
\(V\) is positive on the region \(2x^2 + 8y^2 < 8\) with \(x > 0\) and \(y > 0\) and \(V = 0\) on the boundary.

Since there is only one critical point, it must be a maximum.
Note: Problem 12 shows \(V\) is a local maximum.

16. If two adjustable resistors, with resistances \(R_1\) and \(R_2\), are arranged in parallel, the total resistance \(R\) is given by
\[R = \frac{R_1 R_2}{R_1 + R_2}\]
Currently, \(R_1 = 3\Omega\) and \(R_2 = 7\Omega\) and they are changing according to \(\frac{dR_1}{dt} = -0.1\) \(\frac{\Omega}{\text{sec}}\) and \(\frac{dR_2}{dt} = 0.2\ \frac{\Omega}{\text{sec}}\). Find \(R\) and \(\frac{dR}{dt}\). Is \(R\) increasing or decreasing?

**SOLUTION:**
\[R = \frac{3 \cdot 7}{3 + 7} = 2.1\Omega\]
\[\frac{dR}{dt} = \frac{\partial R}{\partial R_1} \frac{dR_1}{dt} + \frac{\partial R}{\partial R_2} \frac{dR_2}{dt} = \frac{(R_1 + R_2)R_2 - R_1 R_2 (1)}{(R_1 + R_2)^2} \frac{dR_1}{dt} + \frac{(R_1 + R_2)R_1 - R_1 R_2 (1)}{(R_1 + R_2)^2} \frac{dR_2}{dt}\]
\[= \frac{(R_2)^2}{(R_1 + R_2)^2} \frac{dR_1}{dt} + \frac{(R_1)^2}{(R_1 + R_2)^2} \frac{dR_2}{dt} = \frac{7^2}{(3 + 7)^2} (-0.1) + \frac{3^2}{(3 + 7)^2} (0.2) = -0.9 + 1.8\]
\[= 0.9\ \frac{\Omega}{\text{sec}}\]
So \(R\) is decreasing.