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MATH 251 Exam 2 Version B Spring 2013

Sections 506 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. Compute $\iint\limits_R xy\,dA$ over the region R between the parabola $y = x^2$ and the line $y = 2x$.

a. $\dfrac{29}{6}$

b. $\dfrac{10}{3}$

c. $\dfrac{32}{15}$

d. $\dfrac{8}{3}$ Correct Choice

e. $\dfrac{4}{3}$

SOLUTION: $x^2 = 2x \qquad x = 0, 2$

$$\iint\limits_R xy\,dA = \int_0^2 \int_{x^2}^{2x} xy\,dy\,dx = \int_0^2 x\left[\frac{y^2}{2}\right]_{x^2}^{2x} dx = \frac{1}{2}\int_0^2 x(4x^2 - x^4)\,dx = \frac{1}{2}\int_0^2 (4x^3 - x^5)\,dx$$

$$= \frac{1}{2}\left[x^4 - \frac{x^6}{6}\right]_0^2 = \frac{1}{2}\left(16 - \frac{32}{3}\right) = \frac{8}{3}$$

2. Compute $\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x\,dy\,dx$ by converting to polar coordinates.

a. 3

b. 9

c. 18 Correct Choice

d. 27

e. 36

SOLUTION: The region is a semicircle in the right half plane.

$$\int_0^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x\,dy\,dx = \int_{-\pi/2}^{\pi/2} \int_0^3 r\cos\theta\, r\,dr\,d\theta = \left[\sin\theta\right]_{-\pi/2}^{\pi/2}\left[\frac{r^3}{3}\right]_0^3 = (1 - -1)(9 - 0) = 18$$

3. Find the mass of the triangular plate with vertices $(0,0)$, $(2,-6)$ and $(2,6)$ if the surface density is $\rho = x^2$.

a. 24 Correct Choice

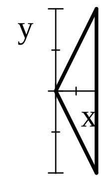
b. 16

c. 12

d. 6

e. 4

SOLUTION: Edges: $y = -3x$ and $y = 3x$



$$M = \iint \rho\, dA = \int_0^2 \int_{-3x}^{3x} x^2\, dy\, dx = \int_0^2 \left[x^2 y \right]_{-3x}^{3x} dx = \int_0^2 6x^3\, dx = \left[\frac{3}{2} x^4 \right]_0^2 = 24$$

4. Find the center of mass of the triangular plate with vertices $(0,0)$, $(2,-6)$ and $(2,6)$ if the surface density is $\rho = x^2$.

a. $\left(\frac{6}{5}, 0\right)$

b. $\left(\frac{5}{9}, 0\right)$

c. $\left(\frac{5}{8}, 0\right)$

d. $\left(\frac{9}{5}, 0\right)$

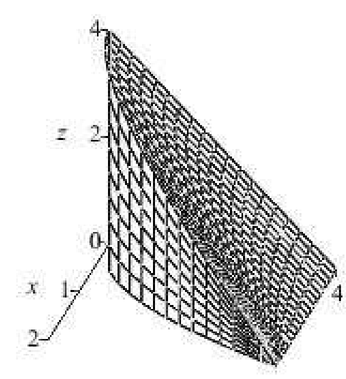
e. $\left(\frac{8}{5}, 0\right)$ Correct Choice

SOLUTION: $\bar{y} = 0$ by symmetry.

$$M_y = \iint x\rho\, dA = \int_0^2 \int_{-3x}^{3x} x^3\, dy\, dx = \int_0^2 \left[x^3 y \right]_{-3x}^{3x} dx = \int_0^2 6x^4\, dx = \left[\frac{6x^5}{5} \right]_0^2 = \frac{192}{5}$$

$$\bar{x} = \frac{M_y}{M} = \frac{192}{5 \cdot 24} = \frac{8}{5}$$

5. Which of the following integrals is NOT equivalent to $\int_0^4 \int_0^{4-z} \int_0^{\sqrt{y}} f(x,y,z)\, dx\, dy\, dz$? The region is shown.



a. $\int_0^4 \int_0^{\sqrt{4-z}} \int_{x^2}^{4-z} f(x,y,z)\, dy\, dx\, dz$

b. $\int_0^4 \int_0^{\sqrt{y}} \int_0^{4-y} f(x,y,z)\, dz\, dx\, dy$

c. $\int_0^4 \int_0^{4-y} \int_{y^2}^{2} f(x,y,z)\, dx\, dz\, dy$ Correct Choice

d. $\int_0^2 \int_{x^2}^{4} \int_0^{4-y} f(x,y,z)\, dz\, dy\, dx$

e. $\int_0^2 \int_0^{4-x^2} \int_{x^2}^{4-z} f(x,y,z)\, dy\, dz\, dx$

SOLUTION: (c) should be $\int_0^4 \int_0^{4-y} \int_0^{\sqrt{y}} f(x,y,z)\, dx\, dz\, dy$

6. Find the area of one petal of the 8 petaled daisy $r = \sin(4\theta)$.

a. $\frac{\pi}{32}$

b. $\frac{\pi}{16}$ Correct Choice

c. $\frac{\pi}{8}$

d. $\frac{\pi}{4}$

e. $\frac{\pi}{2}$

SOLUTION: $r = 0$ when $4\theta = 0, \pi$ or $\theta = 0, \pi/4$

$$A = \iint 1\,dA = \int_0^{\pi/4}\int_0^{\sin(4\theta)} r\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^2}{2}\right]_0^{\sin(4\theta)} d\theta = \frac{1}{2}\int_0^{\pi/4}\sin^2(4\theta)\,d\theta = \frac{1}{2}\int_0^{\pi/4}\frac{1-\cos(8\theta)}{2}\,d\theta$$

$$= \frac{1}{4}\left[\theta - \frac{\sin(8\theta)}{8}\right]_0^{\pi/4} = \frac{\pi}{16}$$

7. Find the mass of the solid between the paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$ if the volume density is $\rho = z$.

a. 4π

b. 8π

c. 16π

d. 32π

e. 64π Correct Choice

SOLUTION: $z = r^2 = 8 - r^2$ $\quad 2r^2 = 8$ $\quad r = 2$

$$M = \iiint \rho\,dV = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} z\,r\,dz\,dr\,d\theta = 2\pi\int_0^2\left[r\frac{z^2}{2}\right]_{z=r^2}^{8-r^2} dr = \pi\int_0^2 r\left[(8-r^2)^2 - r^4\right]dr$$

$$= \pi\int_0^2 64r - 16r^3\,dr = \pi[32r^2 - 4r^4]_0^2 = 64\pi$$

8. Compute $\iiint(x^2+y^2)\,z\,dV$ over the solid hemisphere $0 \le \sqrt{x^2+y^2+z^2} \le 2$

a. $\frac{8}{3}\pi$

b. $\frac{16}{3}\pi$ Correct Choice

c. $\frac{64}{9}\pi$

d. 2π

e. 0

SOLUTION: $(x^2+y^2)z = \left((\rho\sin\varphi\cos\theta)^2 + (\rho\sin\varphi\sin\theta)^2\right)\rho\cos\varphi = \rho^3\sin^2\varphi\cos\varphi \quad dV = \rho^2\sin\varphi$

$$\iiint(x^2+y^2)z\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^2 \rho^3\sin^2\varphi\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\int_0^{\pi/2}\sin^3\varphi\cos\varphi\,d\varphi\int_0^2\rho^5\,d\rho$$

$$= 2\pi\left[\frac{\sin^4\varphi}{4}\right]_0^{\pi/2}\left[\frac{\rho^6}{6}\right]_0^2 = 2\pi\frac{1}{4}\frac{32}{3} = \frac{16}{3}\pi$$

9. Which integral gives the arclength of the ellipse $\vec{r}(\theta) = (6\cos\theta, 3\sin\theta, 3\sin\theta)$?

a. $\int_0^{2\pi} \sqrt{54}\, d\theta$

b. $\int_0^{2\pi} 3\sqrt{2 + 2\sin^2\theta}\, d\theta$ Correct Choice

c. $\int_0^{2\pi} 3\sqrt{2 + 2\cos^2\theta}\, d\theta$

d. $\int_0^{2\pi} 3\sqrt{4 + 2\cos^2\theta}\, d\theta$

e. $\int_0^{2\pi} \sqrt{2}\,(3 + 3\sin^2\theta)\, d\theta$

SOLUTION: $\vec{v} = (-6\sin\theta, 3\cos\theta, 3\cos\theta)$ $|\vec{v}| = \sqrt{36\sin^2\theta + 9\cos^2\theta + 9\cos^2\theta}$

$$L = \oint ds = \int_0^{2\pi} |\vec{v}|\, d\theta = \int_0^{2\pi} 3\sqrt{4\sin^2\theta + 2\cos^2\theta}\, d\theta = \int_0^{2\pi} 3\sqrt{2 + 2\sin^2\theta}\, d\theta$$

10. A helical thermocouple whose shape is the curve $\vec{r}(t) = (3\cos t, 3\sin t, 4t)$ for $0 \le t \le 4\pi$ is placed in a pot of water where the temperature is $T = (41 + x^2 + y^2 + z)$°C. Find the average temperature of the water as measured by the thermocouple.

HINT: $f_{\text{ave}} = \dfrac{\int f ds}{\int ds}$

a. $50 + 8\pi$ Correct Choice

b. $50 + 16\pi$

c. $1000\pi + 160\pi^2$

d. $250 + 40\pi$

e. $\dfrac{173}{4} + 8\pi$

SOLUTION: $\vec{v} = (-3\sin t, 3\cos t, 4)$ $|\vec{v}| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$

$$\int ds = \int_0^{4\pi} 5\, dt = 20\pi \qquad T = 41 + x^2 + y^2 + z = 41 + 9 + 4t = 50 + 4t$$

$$\int T ds = \int_0^{4\pi} (50 + 4t)5\, dt = 5[50t + 2t^2]_0^{4\pi} = 1000\pi + 160\pi^2$$

$$T_{\text{ave}} = \frac{1000\pi + 160\pi^2}{20\pi} = 50 + 8\pi$$

11. Find a scalar potential, $f(x,y,z)$, for $\vec{F}(x,y,z) = (2xy^2 + 2x + 2xz, 2x^2y - 3z, x^2 + 3z^2 - 3y)$. Then compute $f(2,2,2) - f(1,1,1)$.

a. 25

b. 23 Correct Choice

c. 7

d. 1

e. 0

SOLUTION:

$$\partial_x f = 2xy^2 + 2x + 2xz \quad \Rightarrow \quad f = x^2y^2 + x^2 + x^2z + g(y,z)$$
$$\partial_y f = 2x^2y - 3z \quad \Rightarrow \quad f = x^2y^2 - 3yz + h(x,z)$$
$$\partial_z f = x^2 + 3z^2 - 3y \quad \Rightarrow \quad f = x^2z + z^3 - 3yz + k(x,y)$$
$$f(x,y,z) = x^2y^2 + x^2 + x^2z - 3yz + z^3 + C$$
$$f(2,2,2) - f(1,1,1) = (16 + 4 + 8 - 12 + 8) - (1 + 1 + 1 - 3 + 1) = 23$$

12. If $f = x^2 + y^2 - 2z^2$ and $\vec{F} = (xz, yz, -z^2)$, which of the following is false?

a. $\vec{\nabla} \cdot \vec{\nabla} f = 0$

b. $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$

c. $\vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) = \vec{0}$

d. $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$

e. None of the above. They are all true. Correct Choice

SOLUTION: $\vec{\nabla} \times \vec{\nabla} f = \vec{0}$ and $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$ are always true.

$\vec{\nabla} f = (2x, 2y, -4z)$ $\quad \vec{\nabla} \cdot \vec{\nabla} f = 2 + 2 - 4 = 0$

$\vec{\nabla} \cdot \vec{F} = z + z - 2z = 0$ $\quad \vec{\nabla}\left(\vec{\nabla} \cdot \vec{F}\right) = \vec{\nabla}(0) = \vec{0}$

13. Compute $\iiint_C \vec{\nabla} \cdot \vec{G}\, dV$ for $\vec{G} = (xz, yz, z^2)$ over the solid cylinder $x^2 + y^2 \le 25$ with $0 \le z \le 4$.

a. 40π

b. 80π

c. 200π

d. 400π

e. 800π Correct Choice

SOLUTION: $\vec{\nabla} \cdot \vec{G} = z + z + 2z = 4z$

$$\iiint_C \vec{\nabla} \cdot \vec{G}\, dV = \int_0^4 \int_0^{2\pi} \int_0^5 4zr\, dr\, d\theta\, dz = 2\pi\Big[z^2\Big]_0^4 \Big[r^2\Big]_0^5 = 2\pi 16 \cdot 25 = 800\pi$$

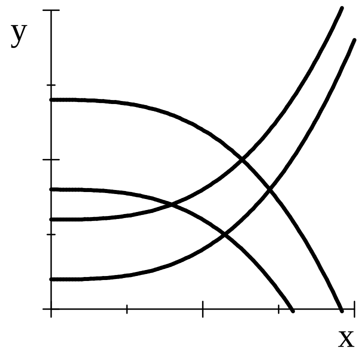
Work Out: (Points indicated. Part credit possible. Show all work.)

14. (12 points) Compute $\iint_D x^2\,dA$ over the "diamond" shaped region bounded by the curves

$y = 1 + x^3, \quad y = 3 + x^3, \quad y = 4 - x^3, \quad y = 7 - x^3.$

HINT: Define curvilinear coordinates (u,v) so that

$y = u + x^3$ and $y = v - x^3$.



a. (2 pts) What are the boundaries in terms of u and v?

SOLUTION: $u = 1, \quad u = 3, \quad v = 4, \quad v = 7$

b. (3 pts) Find formulas for x and y in terms of u and v.

SOLUTION: Add the formulas: $2y = u + v \qquad y = \dfrac{u+v}{2}$

Subtract the formulas: $0 = u - v + 2x^3 \qquad x = \left(\dfrac{v-u}{2}\right)^{1/3}$

c. (4 pts) Find the Jacobian factor $J = \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|$.

SOLUTION:
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{3}\left(\dfrac{v-u}{2}\right)^{-2/3}\left(-\dfrac{1}{2}\right) & \dfrac{1}{2} \\ \dfrac{1}{3}\left(\dfrac{v-u}{2}\right)^{-2/3}\left(\dfrac{1}{2}\right) & \dfrac{1}{2} \end{vmatrix}$$

$$= -\frac{1}{12}\left(\frac{v-u}{2}\right)^{-2/3} - \frac{1}{12}\left(\frac{v-u}{2}\right)^{-2/3} = -\frac{1}{6}\left(\frac{v-u}{2}\right)^{-2/3}$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{6}\left(\frac{v-u}{2}\right)^{-2/3}$$

d. (1 pts) Express the integrand in terms of u and v.

SOLUTION: $x^2 = \left(\dfrac{v-u}{2}\right)^{2/3}$

e. (2 pts) Compute the integral.

SOLUTION:
$$\iint_D x^2\,dA = \iint_D x^2 J\,du\,dv = \int_4^7\int_1^3 \left(\frac{v-u}{2}\right)^{2/3}\frac{1}{6}\left(\frac{v-u}{2}\right)^{-2/3} du\,dv = \frac{1}{6}(2)(3) = 1$$

15. (28 points) Consider the elliptical region, E, in the plane $z = 2 + x + y$ above the circle $x^2 + y^2 \le 4$ oriented upwards.

HINT: This ellipse may be parametrized by $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 2 + r\cos\theta + r\sin\theta)$.

a. (10 pts) Find the normal vector $\vec{N}$ to the ellipse and its length $|\vec{N}|$.

Note: $\vec{N}$ starts hard but simplifies!

SOLUTION:

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & \cos\theta + \sin\theta \;) \\ (-r\sin\theta & r\cos\theta & -r\sin\theta + r\cos\theta) \end{array} \right| \qquad \begin{array}{rl} & \hat{\imath}[\sin\theta(-r\sin\theta + r\cos\theta) - r\cos\theta(\cos\theta + \sin\theta)] \\ \vec{N} = & -\hat{\jmath}[\cos\theta(-r\sin\theta + r\cos\theta) + r\sin\theta(\cos\theta + \sin\theta)] \\ & +\hat{k}[r\cos^2\theta + r\sin^2\theta] \end{array}$$

$\vec{N} = (-r, -r, r)$ Oriented correctly because $N_3 = r \ge 0$. $|\vec{N}| = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r$

b. (3 pts) Find the surface area of the ellipse.

SOLUTION: $0 \le r \le 2 \qquad 0 \le \theta \le 2\pi$

$$A = \iint_E dS = \int\int_E |\vec{N}|\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \sqrt{3}\,r\,dr\,d\theta = 2\pi\sqrt{3}\left[\frac{r^2}{2}\right]_0^2 = 4\pi\sqrt{3}$$

c. (3 pts) Find the mass of the ellipse if the surface density is $\rho = x^2 + y^2$.

SOLUTION: $\rho = x^2 + y^2 = r^2$

$$M = \iint_E \rho\,dS = \int_0^{2\pi}\int_0^2 \sqrt{3}\,r^3\,dr\,d\theta = 2\pi\sqrt{3}\left[\frac{r^4}{4}\right]_0^2 = 8\pi\sqrt{3}$$

d. (12 pts) If $\vec{F} = (-yz, xz, z^2)$, compute the surface integral $\iint_E \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

SOLUTION: $$\vec{\nabla} \times \vec{F} = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{array} \right| = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$(\vec{\nabla} \times \vec{F})(\vec{R}(r,\theta)) = (-r\cos\theta, -r\sin\theta, 4 + 2r\cos\theta + 2r\sin\theta)$ Recall: $\vec{N} = (-r, -r, r)$

$$\vec{\nabla} \times \vec{F} \cdot \vec{N} = r^2\cos\theta + r^2\sin\theta + 4r + 2r^2\cos\theta + 2r^2\sin\theta = 4r + 3r^2\cos\theta + 3r^2\sin\theta$$

$$\iint_E \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{\nabla} \times \vec{F} \cdot \vec{N}\,dr\,d\theta = \int_0^2\int_0^{2\pi} (4r + 3r^2\cos\theta + 3r^2\sin\theta)\,d\theta\,dr$$

$$= \int_0^2 \Big[4r\theta + 3r^2\sin\theta - 3r^2\cos\theta\Big]_{\theta=0}^{2\pi} dr = \int_0^2 4r2\pi\,dr = \Big[4\pi r^2\Big]_0^2 = 16\pi$$

16. (12 points) Compute $\iint_C \vec{G} \cdot d\vec{S}$ for $\vec{G} = (xz, yz, z^2)$ over the cylinder $x^2 + y^2 = 25$ for $0 \le z \le 4$ with outward normal.

SOLUTION: $\vec{R}(\theta, z) = (5\cos\theta, 5\sin\theta, z)$

$$\begin{array}{r} \\ \vec{e}_\theta = \\ \vec{e}_z = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (-5\sin\theta & 5\cos\theta & 0\ ) \\ (\ 0 & 0 & 1\ ) \end{array} \right|$$

$\vec{N} = \hat{\imath}(5\cos\theta) - \hat{\jmath}(-5\sin\theta) + \hat{k}(0) = (5\cos\theta, 5\sin\theta, 0)$

Oriented correctly

$\vec{G}\left(\vec{R}(\theta, z)\right) = (5z\cos\theta, 5z\sin\theta, z^2)$ $\quad \vec{G} \cdot \vec{N} = 25z\cos^2\theta + 25z\sin^2\theta = 25z$

$$\iint_C \vec{G} \cdot d\vec{S} = \iint_C \vec{G} \cdot \vec{N}\, d\theta\, dz = \int_0^4 \int_0^{2\pi} 25z\, d\theta\, dz = 2\pi\, 25 \left[\frac{z^2}{2} \right]_0^4 = 400\pi$$