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MATH 251 Final Exam Spring 2013

Sections 506 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. How much work is done to push a box up an incline from $(2,1)$ to $(4,2)$ by the force $\vec{F} = (3,2)$?

a. 65

b. $\sqrt{65}$

c. $2\sqrt{65}$

d. 64

e. 8 Correct Choice

SOLUTION: $\vec{D} = (4,2) - (2,1) = (2,1)$ $W = \vec{F} \cdot \vec{D} = 6 + 2 = 8$

2. The graph of $2x^2 + 4x + 3y - 3z^2 - 6z = 4$ is a

a. hyperboloid of 1-sheet

b. hyperboloid of 2-sheets

c. elliptic paraboloid

d. hyperbolic paraboloid Correct Choice

e. cone

SOLUTION: y is linear, x^2 and z^2 have opposite signs $\Rightarrow$ hyperbolic paraboloid

3. Find the tangential acceleration, a_T, or the curve $\vec{r}(t) = \left(t^2, \frac{4}{3}t^3, t^4\right)$. HINT: Perfect square.

a. $2 - 12t^2$

b. $2 + 12t^2$ Correct Choice

c. $2t - 4t^3$

d. $2t + 4t^3$

e. $2 + 16t^2$

SOLUTION: $\vec{v} = (2t, 4t^2, 4t^3)$ $|\vec{v}| = \sqrt{4t^2 + 16t^4 + 16t^6} = 2t + 4t^3$ $a_T = \dfrac{d|\vec{v}|}{dt} = 2 + 12t^2$

4. Find the equation of the plane through the points $(1,2,3)$, $(2,4,2)$ and $(-1,2,4)$.
Its z-intercept is

a. 1
b. 2
c. 4 Correct Choice
d. 8
e. 16

SOLUTION: $P = (1,2,3)$ $\vec{u} = (2,4,2) - (1,2,3) = (1,2,-1)$ $\vec{v} = (-1,2,4) - (1,2,3) = (-2,0,1)$

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 0 & 1 \end{vmatrix} = \hat{\imath}(2-0) - \hat{\jmath}(1-2) + \hat{k}(0--4) = (2,1,4)$$

$\vec{N} \cdot X = \vec{N} \cdot P$ $2x + y + 4z = 2(1) + (2) + 4(3) = 16$
The z-intercept is when $x = y = 0$ and $z = 4$.

5. Find the plane tangent to the graph of the function $z = 2x^2y$ at the point $(x,y) = (3,2)$.
Its z-intercept is

a. -72 Correct Choice
b. -36
c. 0
d. 36
e. 72

SOLUTION:

$f(x,y) = 2x^2y$ $f(3,2) = 36$ $z = f_{\tan}(x,y) = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2)$
$f_x(x,y) = 4xy$ $f_x(3,2) = 24$ $= 36 + 24(x-3) + 18(y-2) = 24x + 18y - 72$
$f_y(x,y) = 2x^2$ $f_y(3,2) = 18$ z-intercept $= -72$

6. Find the plane tangent to the level set of the function $F(x,y,z) = 2x^2yz^3$ at the point $(x,y,z) = (3,2,1)$. Its z-intercept is

a. 1
b. 2 Correct Choice
c. 3
d. 108
e. 216

SOLUTION: $\vec{\nabla}F = (4xyz^3, 2x^2z^3, 6x^2yz^2)$ $\vec{N} = \vec{\nabla}F(3,2,1) = (24,18,108)$
$\vec{N} \cdot X = \vec{N} \cdot P$ $24x + 18y + 108z = 24 \cdot 3 + 18 \cdot 2 + 108 \cdot 1 = 216$
z-intercept $= 216/108 = 2$

7. The volume of a cone is $V = \frac{1}{3}\pi r^2 h$.
If the radius r is currently 3 cm and decreasing at 2 cm/sec
while the height h is currently 4 cm and increasing at 1 cm/sec,
is the volume increasing or decreasing and at what rate?

a. increasing at 19π cm$^3/\text{sec}$

b. increasing at 13π cm$^3/\text{sec}$

c. neither increasing nor decreasing

d. decreasing at 13π cm$^3/\text{sec}$ Correct Choice

e. decreasing at 19π cm$^3/\text{sec}$

SOLUTION: $\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = \frac{2}{3}\pi rh\frac{dr}{dt} + \frac{1}{3}\pi r^2\frac{dh}{dt} = \frac{2}{3}\pi(3)(4)(-2) + \frac{1}{3}\pi(3)^2(1) = -13\pi$
This is negative and so decreasing.

8. Duke Skywater is traveling in the Millenium Eagle through a dangerous galactic politon field whose density is $\rho = 2x^2yz^3$. If Duke's current position and velocity are $\vec{r} = (3,2,1)$ and $\vec{v} = (.25,.5,-.25)$, what is the current time rate of change of the politon field as seen by Duke?

a. -12 Correct Choice

b. -120

c. 12

d. 120

e. $12,564$

SOLUTION: $\vec{\nabla}\rho = (4xyz^3, 2x^2z^3, 6x^2yz^2)$ $\vec{\nabla}\rho(3,2,1) = (24,18,108)$
$\frac{d\rho}{dt} = \vec{v} \cdot \vec{\nabla}\rho = (.25,.5,-.25) \cdot (24,18,108) = 6 + 9 - 27 = -12$

9. Duke Skywater is traveling in the Millenium Eagle through a dangerous galactic politon field whose density is $\rho = 2x^2yz^3$. If Duke's current position is $\vec{r} = (3,2,1)$, in what unit vector direction should he travel to **reduce** the politon field as fast as possible?

a. $\frac{1}{\sqrt{349}}(4,3,18)$

b. $\frac{1}{\sqrt{349}}(4,-3,18)$

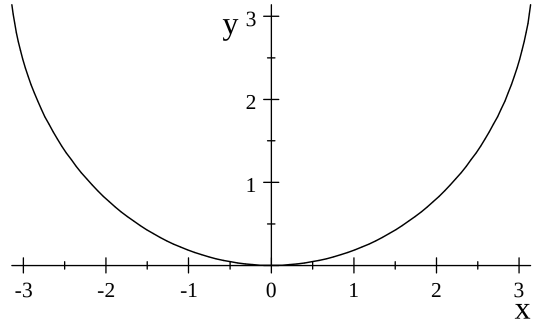
c. $\frac{1}{\sqrt{349}}(-4,-3,-18)$ Correct Choice

d. $\frac{1}{\sqrt{349}}(-4,3,-18)$

SOLUTION: $\vec{\nabla}\rho = (4xyz^3, 2x^2z^3, 6x^2yz^2)$ $\vec{\nabla}\rho(3,2,1) = (24,18,108)$
The politon field decreases fastest in the direction of $-\vec{\nabla}\rho = -6(4,3,18)$. $\sqrt{4^2 + 3^2 + 18^2} = \sqrt{349}$
So the unit vector direction is $\hat{u} = \frac{-1}{|\vec{\nabla}\rho|}\vec{\nabla}\rho = \frac{1}{\sqrt{349}}(-4,-3,-18)$

10. Compute $\int \vec{F} \cdot d\vec{s}$ where $\vec{F} = (2x+2y, 2x+2y)$ along the curve $\vec{r}(t) = \left(4\sqrt{2}\,t\cos(t), 4\sqrt{2}\,t\sin(t)\right)$ for $-\frac{\pi}{4} \le t \le \frac{\pi}{4}$.

HINT: Find a scalar potential.



a. 2π

b. 4π

c. 8π

d. $4\pi^2$ Correct Choice

e. $8\pi^2$

SOLUTION: $\vec{\nabla}f = \vec{F} = (2x+2y, 2x+2y)$

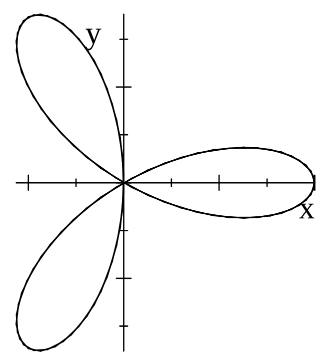
$$\partial_x f = 2x+2y \Rightarrow f = x^2 + 2xy + g(y)$$
$$\partial_y f = 2x+2y \Rightarrow f = y^2 + 2xy + h(x)$$

$f = x^2 + y^2 + 2xy$ $\quad \vec{r}\left(\pm\frac{\pi}{4}\right) = \left(4\sqrt{2}\,\frac{\pi}{4}\frac{1}{\sqrt{2}}, \pm 4\sqrt{2}\,\frac{\pi}{4}\frac{1}{\sqrt{2}}\right) = (\pi, \pm\pi)$ By the F.T.C.C.

$$\int \vec{F} \cdot d\vec{s} = \int_{(\pi,-\pi)}^{(\pi,\pi)} \vec{\nabla}f \cdot d\vec{s} = f(\pi,\pi) - f(\pi,-\pi) = (\pi^2 + \pi^2 + 2\pi^2) - (\pi^2 + \pi^2 - 2\pi^2) = 4\pi^2$$

11. Compute $\oint \vec{F} \cdot d\vec{s}$ for $\vec{F} = (x-y, x+y)$ counterclockwise around the one leaf of the 3 leaf rose $r = \cos(3\theta)$ with $x \ge 0$.

HINT: Use Green's Theorem.



a. $\frac{\pi}{2}$

b. $\frac{\pi}{3}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{6}$ Correct Choice

e. $\frac{\pi}{12}$

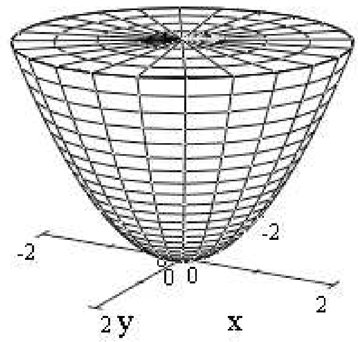
SOLUTION: $\cos(3\theta) = 0$ when $3\theta = \pm\frac{\pi}{2}$ or $\theta = \pm\frac{\pi}{6}$ Let $P = x - y$ and $Q = x + y$.

By Green's Theorem,

$$\oint \vec{F} \cdot d\vec{s} = \oint P\,dx + Q\,dy = \iint (\partial_x Q - \partial_y P)\,dx\,dy = \iint (1 - -1)\,dx\,dy = \int_{-\pi/6}^{\pi/6} \int_0^{\cos(3\theta)} 2r\,dr\,d\theta$$

$$= \int_{-\pi/6}^{\pi/6} \left[r^2\right]_0^{\cos(3\theta)} d\theta = \int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\,d\theta = \int_{-\pi/6}^{\pi/6} \frac{1+\cos(6\theta)}{2}\,d\theta = \frac{1}{2}\left[\theta + \frac{\sin(6\theta)}{6}\right]_{-\pi/6}^{\pi/6} = \frac{\pi}{6}$$

12. Compute $\iint \vec{F} \cdot d\vec{S}$ for $\vec{F} = (xy^2, yx^2, z(x^2+y^2))$
over the complete surface of the solid
above the paraboloid $z = x^2 + y^2$
below the plane $z = 4$, oriented outward.



a. $\frac{64}{5}\pi$

b. $\frac{64}{3}\pi$ Correct Choice

c. $\frac{64}{15}\pi$

d. $\frac{256}{15}\pi$

e. $\frac{256}{3}\pi$

SOLUTION: By Gauss' Theorem $\iint \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{F}\, dV$ Use cylindrical coordinates.

$\vec{\nabla} \cdot \vec{F} = y^2 + x^2 + x^2 + y^2 = 2r^2 \qquad dV = r\,dr\,d\theta\,dz$

$$\iint \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2\int_{r^2}^4 2r^3\,dz\,dr\,d\theta = 2\pi\int_0^2\Big[2r^3z\Big]_{r^2}^4 dr = 2\pi\int_0^2 (8r^3 - 2r^5)\,dr = 2\pi\Big[2r^4 - \frac{r^6}{3}\Big]_0^2$$
$$= 2\pi\left(32 - \frac{64}{3}\right) = \frac{64}{3}\pi$$

13. Compute $\iint (x^3 + xy^2)\,dx\,dy$ over the quarter circle $x^2 + y^2 \le 4$ in the first quadrant.

a. 4

b. $\frac{8}{5}\pi$

c. $\frac{16}{5}\pi$

d. $\frac{64}{5}\pi$

e. $\frac{32}{5}$ Correct Choice

SOLUTION: $x^3 + xy^2 = x(x^2+y^2) = r^3\cos\theta \qquad dx\,dy = r\,dr\,d\theta$

$$\iint x^3 + xy^2\,dx\,dy = \int_0^{\pi/2}\int_0^2 r^4\cos\theta\,dr\,d\theta = \Big[\sin\theta\Big]_0^{\pi/2}\Big[\frac{r^5}{5}\Big]_0^2 = \frac{32}{5}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (15 points) The half cylinder $x^2 + y^2 = 9$ for $y \ge 0$ and $0 \le z \le 4$ has mass surface density $\rho = y^2$. Find the total mass and the center of mass. Follow these steps:

Parametrize the surface:

$$\vec{R}(\theta, z) = (3\cos\theta, 3\sin\theta, z)$$

Find the tangent vectors, the normal vector and its length:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_\theta = & (-3\sin\theta & 3\cos\theta, & 0) \\ \vec{e}_z = & (-0, & 0, & 1) \end{vmatrix} \qquad \begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta \\ &= \hat{\imath}(3\cos\theta) - \hat{\jmath}(-3\sin\theta) + \hat{k}(0) \\ &= (3\cos\theta, 3\sin\theta, 0) \end{aligned}$$

$$\left|\vec{N}\right| = \sqrt{9\cos^2\theta + 9\sin^2\theta} = 3$$

Evaluate the density on the surface and compute the total mass:

$$\rho = y^2 \qquad \rho|_{\vec{R}(\theta,z)} = 9\sin^2\theta$$

$$M = \iint \rho\, dS = \iint \rho|_{\vec{R}(\theta,z)} \left|\vec{N}\right| d\theta\, dz = \int_0^4 \int_0^\pi 9\sin^2\theta\, 3\, d\theta\, dz = 27 \cdot 4 \int_0^\pi \frac{1 - \cos(2\theta)}{2}\, d\theta$$

$$= 54\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^\pi = 54\pi$$

Use symmetry to determine the x and z components fo the center of mass and then compute the y component of the center of mass.

$$\bar{x} = 0 \qquad \bar{z} = 2$$

$$M_{xz} = \iint y\rho\, dS = \int_0^4 \int_0^\pi 27\sin^3\theta\, 3\, d\theta\, dz = 81 \cdot 4 \int_0^\pi (1 - \cos^2\theta)\sin\theta\, d\theta \qquad u = \cos\theta \quad du = -\sin\theta\, d\theta$$

$$= -324 \int_1^{-1} (1 - u^2)\, du = -324\left[u - \frac{u^3}{3}\right]_1^{-1} = -324\left[-1 - \frac{-1}{3}\right] + 324\left[1 - \frac{1}{3}\right] = 648 \cdot \frac{2}{3} = 432$$

$$\bar{y} = \frac{M_{xz}}{M} = \frac{432}{54\pi} = \frac{8}{\pi}$$

15. (15 points) A rectangular box sits on the xy-plane with its top 4 vertices on the paraboloid $z + 2x^2 + 8y^2 = 8$. Find the dimensions and volume of the largest such box.

NOTE: Full Credit for solving by Lagrange Multipliers, Half Credit for Eliminating a Variable.

SOLUTION: Let the corner in the first octant be (x,y,z). So x, y and z are positive.
The dimensions are $L = 2x$, $W = 2y$, $H = z$ and the volume is $V = (2x)(2y)z = 4xyz$.
So we need to maximize $V = 4xyz$ subject to the constraint $g = z + 2x^2 + 8y^2 = 8$.

Lagrange Multiplier Method: $\vec{\nabla}V = (4yz, 4xz, 4xy)$ $\vec{\nabla}g = (4x, 16y, 1)$ $\vec{\nabla}V = \lambda\vec{\nabla}g$

$4yz = \lambda 4x \Rightarrow 4xyz = \lambda 4x^2$

$4xz = \lambda 16y \Rightarrow 4xyz = \lambda 16y^2 \Rightarrow 4x^2 = 16y^2 = z \Rightarrow x = \dfrac{\sqrt{z}}{2}$ and $y = \dfrac{\sqrt{z}}{4}$

$4xy = \lambda \Rightarrow 4xyz = \lambda z$

Use the constraint: $z + 2x^2 + 8y^2 = z + \dfrac{z}{2} + \dfrac{z}{2} = 8 \Rightarrow z = 4,\ x = \dfrac{\sqrt{z}}{2} = 1,\ y = \dfrac{\sqrt{z}}{4} = \dfrac{1}{2}$

Eliminate a Variable Method: $z = 8 - 2x^2 - 8y^2$

$V = 4xyz = 4xy(8 - 2x^2 - 8y^2) = 32xy - 8x^3y - 32xy^3$

$V_x = 32y - 24x^2y - 32y^3 = 8y(4 - 3x^2 - 4y^2) = 0 \Rightarrow$ Since $y \neq 0$, $3x^2 + 4y^2 = 4$ (1)

$V_y = 32x - 8x^3 - 96xy^2 = 8x(4 - x^2 - 12y^2) = 0 \Rightarrow$ Since $x \neq 0$, $x^2 + 12y^2 = 4$ (2)

$3 \times (1) - (2):$ $8x^2 = 8 \Rightarrow x = 1$

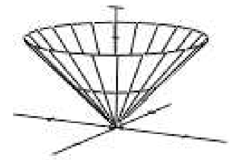
$3 \times (2) - (1):$ $32y^2 = 8 \Rightarrow y = \dfrac{1}{2} \Rightarrow z = 8 - 2x^2 - 8y^2 = 8 - 2 - 2 = 4$

Conclusions: $L = 2x = 2.$ $W = 2y = 1,$ $H = z = 4$ $V = 2 \cdot 1 \cdot 4 = 8$

V is positive on the region $2x^2 + 8y^2 < 8$ with $x > 0$ and $y > 0$ and $V = 0$ on the boundary.
Since there is only one critical point, it must be a maximum.

16. (20 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the cone C given by $z = \sqrt{x^2 + y^2}$ for $z \le 3$



oriented down and out, and the vector field $\vec{F} = (-yz, xz, z^2)$.

Note: The boundary of the cone is the circle, $x^2 + y^2 = 9$,

Be sure to check the orientations. Use the following steps:

a. The cone, C, may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

Compute the surface integral by successively finding:

$\vec{e}_r$, $\vec{e}_\theta$, $\vec{N}$, $\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)}$, $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array}\left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta, & 1) \\ (-r\sin\theta, & r\cos\theta, & 0) \end{array}\right| \qquad \begin{aligned} \vec{N} &= \vec{e}_r \times \vec{e}_\theta \\ &= \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) \\ &= (-r\cos\theta, -r\sin\theta, r) \end{aligned}$$

Reverse $\vec{N} = (r\cos\theta, r\sin\theta, -r)$ so it points down and out.

$$\vec{\nabla} \times \vec{F} = \left|\begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -yz, & xz, & z^2 \end{array}\right| = \hat{\imath}(0 - x) - \hat{\jmath}(0 - -y) + \hat{k}(z - -z) = (-x, -y, 2z)$$

$$\vec{\nabla} \times \vec{F}\Big|_{\vec{R}(r,\theta)} = (-r\cos\theta, -r\sin\theta, 2r)$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^3 (-r^2\cos^2\theta - r^2\sin^2\theta - 2r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^3 (-3r^2)\, dr\, d\theta = 2\pi\Big[-r^3\Big]_0^3 = -54\pi$$

b. Parametrize the circle, ∂C, and compute the line integral by successively finding:

$\vec{r}(\theta)$, $\vec{v}$, $\vec{F}\Big|_{\vec{r}(\theta)}$, $\int_{\partial C} \vec{F} \cdot d\vec{s}$

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 3)$ $\quad \vec{v} = (-3\sin\theta, 3\cos\theta, 0)$ $\quad$ Reverse $\vec{v} = (3\sin\theta, -3\cos\theta, 0)$

$$\vec{F}\Big|_{\vec{r}(\theta)} = (-yz, xz, z^2) = (-9\sin\theta, 9\cos\theta, 9)$$

$$\int_S \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (-27\sin^2\theta - 27\cos^2\theta)\, d\theta = -54\pi$$