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MATH 251 Exam 2 Fall 2014

Sections 508 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Compute $\int_0^{\pi}\int_0^{\theta} r^2\,dr\,d\theta$.

a. $\frac{1}{3}\pi\theta^3$

b. $\frac{1}{6}\pi^3$

c. $\frac{1}{12}\pi^4$ Correct Choice

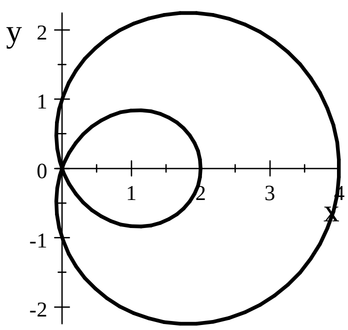
d. $\frac{1}{20}\pi^5$

e. $\frac{4}{3}\pi^4$

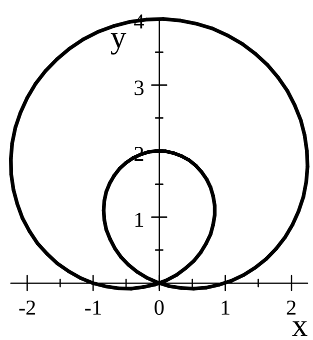
Solution: $\int_0^{\pi}\int_0^{\theta} r^2\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^3}{3}\right]_{r=0}^{\theta} d\theta = \int_0^{\pi}\frac{\theta^3}{3}\,d\theta = \left[\frac{\theta^4}{12}\right]_0^{\pi} = \frac{\pi^4}{12}$

2. Which of the following is the polar plot of $r = 2 + 3\cos(\theta)$?

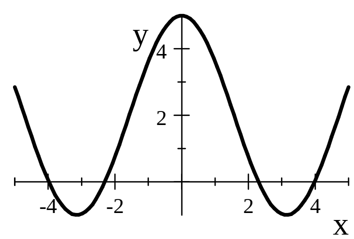
a.



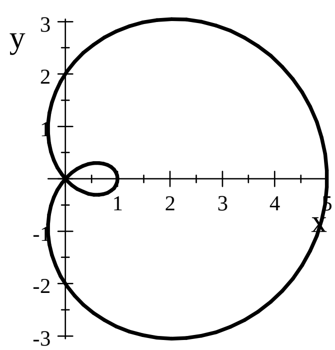
b.



c.

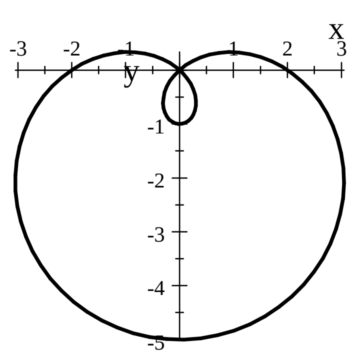


d.



Correct Choice

e.



Solution: c is the rectangular plot not the polar plot. Notice when $\theta = 0$, $r = 1$, which starts the inner loop at $x = 1$.

3. A plate has the shape between the parabola $y = x^2$ and the line $y = 4$.
Find its mass if its surface density is $\rho = y$.

a. $\frac{128}{5}$ Correct Choice
b. $\frac{64}{5}$
c. $\frac{32}{5}$
d. $\frac{16}{5}$
e. $\frac{8}{5}$

Solution: $M = \iint \rho\, dA = \int_{-2}^{2}\int_{x^2}^{4} y\, dy\, dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_{y=x^2}^{4} dx = \int_{-2}^{2}\left(8 - \frac{x^4}{2}\right) dx = \left[8x - \frac{x^5}{10}\right]_{-2}^{2}$

$= 2\left[16 - \frac{16}{5}\right] = 32\left(1 - \frac{1}{5}\right) = \frac{128}{5}$

4. A plate has the shape between the parabola $y = x^2$ and the line $y = 4$.
Find the y-component of its center of mass if its surface density is $\rho = y$.

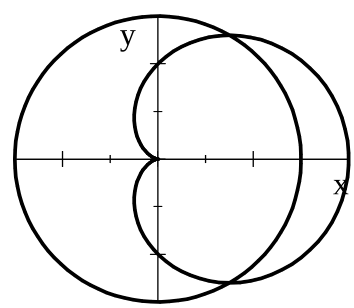
a. $\frac{1024}{7}$
b. $\frac{512}{7}$
c. $\frac{7}{512}$
d. $\frac{20}{7}$ Correct Choice
e. $\frac{10}{7}$

Solution: $M_x = \iint y\rho\, dA = \int_{-2}^{2}\int_{x^2}^{4} y\, y\, dy\, dx = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{y=x^2}^{4} dx = \frac{1}{3}\int_{-2}^{2}(64 - x^6)\, dx = \frac{1}{3}\left[64x - \frac{x^7}{7}\right]_{-2}^{2}$

$= \frac{2}{3}\left[128 - \frac{128}{7}\right] = \frac{256}{3}\left(1 - \frac{1}{7}\right) = \frac{256}{3} \cdot \frac{6}{7} = \frac{512}{7}$

$\bar{y} = \frac{M_x}{M} = \frac{512}{7} \cdot \frac{5}{128} = \frac{20}{7}$

5. Find the mass of the region outside the circle $r = 3$
and inside the cardioid $r = 2 + 2\cos\theta$ if the linear
mass density is $\rho = \frac{1}{r}$.



a. $2 + \frac{2}{3}\pi$
b. $2 - \frac{2}{3}\pi$
c. $2\sqrt{3} - \frac{2\pi}{3}$ Correct Choice
d. $3\sqrt{3} + 2\pi$
e. $3\sqrt{3} - 2\pi$

Solution: Find the intersection points: $2 + 2\cos\theta = 3$ $\quad \cos\theta = \frac{1}{2}$ $\quad \theta = \pm\frac{\pi}{3}$

$M = \iint \rho\, dA = \int_{-\pi/3}^{\pi/3}\int_{3}^{2+2\cos\theta} \frac{1}{r}\, r\, dr\, d\theta = \int_{-\pi/3}^{\pi/3}\left[r\right]_{r=3}^{2+2\cos\theta} d\theta = \int_{-\pi/3}^{\pi/3}(2 + 2\cos\theta - 3)\, d\theta$

$= \left[2\sin\theta - \theta\right]_{-\pi/3}^{\pi/3} = 2\left(2\sin\frac{\pi}{3} - \frac{\pi}{3}\right) = 2\sqrt{3} - \frac{2}{3}\pi$

6. If $\vec{F} = (xyz, 2xyz, 3xyz)$, then $\vec{\nabla} \times \vec{F} =$

a. $(3xz - 2xy, xy - 3yz, 2yz - xz)$ Correct Choice

b. $(3xz - 2xy, 3yz - xy, 2yz - xz)$

c. $yz + 2xz + 3xy$

d. $yz - 2xz + 3xy$

e. 0

Solution: $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ xyz & 2xyz & 3xyz \end{vmatrix} = \hat{\imath}(3xz - 2xy) - \hat{\jmath}(3yz - xy) + \hat{k}(2yz - xz)$

7. If $\vec{F} = (xyz, 2xyz, 3xyz)$, then $\vec{\nabla} \cdot \vec{F} =$

a. $(3xz - 2xy, xy - 3yz, 2yz - xz)$

b. $(3xz - 2xy, 3yz - xy, 2yz - xz)$

c. $yz + 2xz + 3xy$ Correct Choice

d. $yz - 2xz + 3xy$

e. 0

Solution: $\vec{\nabla} \cdot \vec{F} = \partial_x F_1 + \partial_y F_2 + \partial_z F_3 = yz + 2xz + 3xy$

8. Find the average value of $f = z$ on the solid hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$.
Note: The average value of a function on a solid is $f_{\text{ave}} = \dfrac{1}{V} \iiint_V f\,dV$.

a. $\dfrac{3\pi}{8}$

b. $\dfrac{\pi}{2}$

c. $\dfrac{3}{2}$

d. $\dfrac{1}{2}$

e. $\dfrac{9}{8}$ Correct Choice

Solution: $V = \dfrac{1}{2}\dfrac{4}{3}\pi r^3 = \dfrac{2}{3}\pi 3^3 = 18\pi$. In spherical coordinates, $f = z = \rho\cos\varphi$.

$$\iiint_V f\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^3 \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2}\left[\frac{\rho^4}{4}\right]_0^3 = \frac{3^4\pi}{4}$$

$$f_{\text{ave}} = \frac{1}{V}\iiint_V f\,dV = \frac{1}{18\pi} \cdot \frac{3^4\pi}{4} = \frac{9}{8}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

9. (20 points) A rectangular solid box is sitting on the xy-plane with its upper 4 vertices on the paraboloid $z = 16 - x^2 - 4y^2$. Find the dimensions and volume of the largest such box.
Full credit for solving by Lagrange multipliers.
Half credit for solving by Eliminating a Variable.
50% extra credit for solving both ways.

Solution: Maximize: $V = 4xyz$. Note: $x \neq 0 \quad y \neq 0 \quad z \neq 0$ so the volume will not be zero.

Method 1: Lagrange Multipliers:

The constraint is $g = z + x^2 + 4y^2 = 16$

$\vec{\nabla} V = \langle 4yz, 4xz, 4xy \rangle \qquad \vec{\nabla} g = \langle 2x, 8y, 1 \rangle \qquad \vec{\nabla} V = \lambda \vec{\nabla} g$

$4yz = \lambda 2x \qquad 4xz = \lambda 8y \qquad 4xy = \lambda$

$\Rightarrow \quad 4yz = 4xy2x \quad \Rightarrow \quad z = 2x^2 \quad \Rightarrow \quad x^2 = \dfrac{z}{2}$

$\Rightarrow \quad 4xz = 4xy8y \quad \Rightarrow \quad z = 8y^2 \quad \Rightarrow \quad y^2 = \dfrac{z}{8}$

$g = z + \dfrac{z}{2} + \dfrac{z}{2} = 16 \quad \Rightarrow \quad 2z = 16 \quad \Rightarrow \quad z = 8$

$x^2 = 4 \qquad x = 2, \qquad y^2 = 1 \qquad y = 1$

$L = 4, \qquad W = 2, \qquad H = 8, \qquad V = 4xyz = LWH = 4 \cdot 2 \cdot 1 \cdot 8 = 64$

Method 2: Eliminate a Variable:

$V = 4xyz = 4xy(16 - x^2 - 4y^2) = 64xy - 4x^3y - 16xy^3$

$V_x = 64y - 12x^2y - 16y^3 = 0 \quad \Rightarrow \quad 3x^2 + 4y^2 = 16 \quad \Rightarrow \quad 9x^2 + 12y^2 = 48$

$V_y = 64x - 4x^3 - 48xy^2 = 0 \quad \Rightarrow \quad x^2 + 12y^2 = 16$

$\Rightarrow \quad 8x^2 = 32 \quad \Rightarrow \quad x = 2, \qquad 4 + 12y^2 = 16 \quad \Rightarrow \quad y = 1$

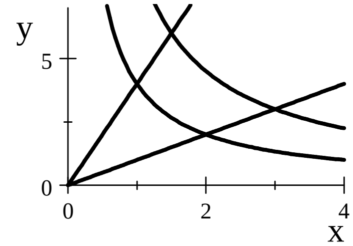
$z = 16 - x^2 - 4y^2 = 16 - 4 - 4 = 8$

$L = 4, \qquad W = 2, \qquad H = 8, \qquad V = 4xyz = LWH = 4 \cdot 2 \cdot 1 \cdot 8 = 64$

10. (20 points) Compute the integral $\iint y\,dA$ over the region in the first quadrant bounded by

$y = x$, $y = 4x$, $y = \frac{4}{x}$, and $y = \frac{9}{x}$.

Use the following steps:



a. Define the curvilinear coordinates u and v by $y = u^2 x$ and $y = \frac{v^2}{x}$.
Express the coordinate system as a position vector.

$$u^2 x = \frac{v^2}{x} \qquad x = \frac{v}{u} \qquad y = u^2 x = u^2 \frac{v}{u} = uv$$

$$\vec{r}(u,v) = (x(u,v), y(u,v)) = \left(\frac{v}{u}, uv\right)$$

b. Find the coordinate tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{r}}{\partial u} = \left(\frac{-v}{u^2}, v\right)$$

$$\vec{e}_v = \frac{\partial \vec{r}}{\partial v} = \left(\frac{1}{u}, u\right)$$

c. Compute the Jacobian factor:

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|-\frac{v}{u} - \frac{v}{u}\right| = \left|\frac{-2v}{u}\right| = \frac{2v}{u}$$

d. Compute the integral:

$$\iint y\,dA = \int_2^3 \int_1^2 uv \frac{2v}{u}\,du\,dv = \int_1^2 1\,du \int_2^3 2v^2\,dv = [u]_1^2 \left[\frac{2v^3}{3}\right]_2^3 = \frac{38}{3}$$

11. (20 points) Compute the flux $\iint_C \vec{F} \cdot d\vec{S}$ of the vector field $\vec{F} = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}, \frac{z}{x^2+y^2} \right)$ outward through the piece of the cylinder $x^2 + y^2 = 9$ between the planes $z = -3$ and $z = x$. Use the following steps:

a. Parametrize the cylinder:

$$\vec{R}(z,\theta) = (3\cos\theta, \quad 3\sin\theta, \quad z)$$

b. Find the tangent vectors:

$$\vec{e}_z = (\quad 0, \quad 0, \quad 1)$$

$$\vec{e}_\theta = (-3\sin\theta, \quad 3\cos\theta, \quad 0)$$

c. Find the normal vector:

$$\vec{N} = \hat{\imath}(-3\cos\theta) - \hat{\jmath}(--3\sin\theta) + \hat{k}(0) = (-3\cos\theta, -3\sin\theta, 0)$$

d. Fix the orientation of the normal (if necessary):

$$\vec{N} = (3\cos\theta, 3\sin\theta, 0)$$

e. Evaluate the vector field on the cylinder:

$$\vec{F}\left(\vec{R}(z,\theta)\right) = \left(\frac{3\cos\theta}{9}, \frac{3\sin\theta}{9}, \frac{z}{9} \right) = \left(\frac{\cos\theta}{3}, \frac{\sin\theta}{3}, \frac{z}{9} \right)$$

f. Evaluate the boundaries on the cylinder:

$z = -3: \qquad z = -3$

$z = x: \qquad z = 3\cos\theta$

g. Calculate the flux:

$$\iint_C \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{N}\, dz\, d\theta = \int_0^{2\pi} \int_{-3}^{3\cos\theta} (\cos^2\theta + \sin^2\theta)\, dz\, d\theta = \int_0^{2\pi} \int_{-3}^{3\cos\theta} 1\, dz\, d\theta$$

$$= \int_0^{2\pi} \left[z \right]_{z=-3}^{3\cos\theta} d\theta = \int_0^{2\pi} (3\cos\theta - -3)\, d\theta = [3\sin\theta + 3\theta]_0^{2\pi} = 6\pi$$