Name________________________

1-12	/60
13	/16
14	/12
15	/12
Total	/100

MATH 251 Exam 1A Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. If $\vec{a} = (4,-2,1)$ and $\vec{b} = (2,-1,1)$, then $\left|\vec{a} - 3\vec{b}\right| =$

a. 1

b. 3 Correct

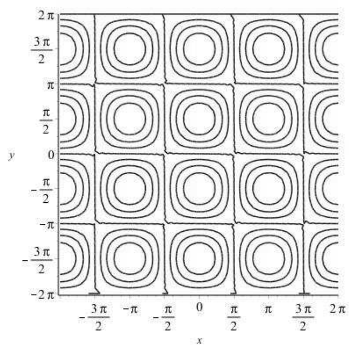
c. 5

d. 9

e. 13

Solution: $\left|\vec{a} - 3\vec{b}\right| = |(-2,1,-2)| = \sqrt{4+1+4} = 3$

2. The plot at the right is the contour plot of which function?
HINT: Where is the level set with value 0?



a. $\sin(x)\cos(y)$

b. $\sin(x)\sin(y)$

c. $\cos(x)\cos(y)$

d. $\cos(x)\sin(y)$ Correct

e. $\sin(xy)$

Solution: There is a level set at $x = \frac{\pi}{2}$ but not $x = 0$. So it must have a $\cos(x)$ but not $\sin(x)$ or $\sin(xy)$ factor. There is a level set at $y = 0$ but not $y = \frac{\pi}{2}$. So it must have a $\sin(y)$ but not $\cos(y)$ factor.

3. Suppose $proj_{\vec{v}}\vec{u} = (3,1)$. Which of the following is **inconsistent** with this fact?

a. $proj_{\perp\vec{v}}\vec{u} = (2,-6)$
b. $proj_{\perp\vec{v}}\vec{u} = (-2,6)$
c. $\vec{u} = (4,-2)$
d. $\vec{v} = (6,2)$
e. $\vec{v} = (1,-3)$ Correct

Solution: $proj_{\vec{v}}\vec{u}$ must be perpendicular to $proj_{\perp\vec{v}}\vec{u}$, and $(2,-6)\cdot(3,1) = 0$ and $(-2,6)\cdot(3,1) = 0$. $proj_{\vec{v}}\vec{u}$ must be parallel to $\vec{v}$, and $(6,2)$ is a multiple but $(1,-3)$ is not.

4. Which of the following is an ellipse in the 1st quadrant tangent to both the x and y-axes?

a. $9(x-3)^2 + 4(y-2)^2 = 36$
b. $4(x-3)^2 + 9(y-2)^2 = 36$ Correct
c. $4(x-2)^2 + 9(y-3)^2 = 36$
d. $4(x-3)^2 + 9(y-2)^2 = 1$
e. $9(x-2)^2 + 4(y-3)^2 = 1$

Solution: The centers are $(3,2)$ in (a), (b) and (d) and $(2,3)$ in (c) and (e). In (d) and (e) the x and y radii are $\frac{1}{2}$ and $\frac{1}{3}$, not enough to reach the axes from the center. In (a) the x radius is 2 which cannot reach the y axis from the center $(3,2)$. In (c) the y radius is 2 which cannot reach the x axis from the center $(2,3)$. In (b) the x radius is 3 and the y radius is 2 just matching the distances from the center $(3,2)$ to the axes.

5. In 3-dimensional space, the equation $x^2 - 4x - y^2 + 6y + z^2 = 5$ is

a. a hyperboloid with center $(2,3,0)$ and axis $\vec{r}(t) = (2,3,t)$.
b. a hyperboloid with center $(2,3,0)$ and axis $\vec{r}(t) = (2,3+t,0)$.
c. a hyperbolic cylinder with axis $\vec{r}(t) = (2,3,t)$.
d. a cone with vertex $(2,3,0)$ and axis $\vec{r}(t) = (2,3+t,0)$. Correct
e. two planes which intersect at the line $\vec{r}(t) = (2,3+t,0)$.

Solution: Complete the squares to get $(x-2)^2 - (y-3)^2 + z^2 = 0$.
This is a cone with axis parallel to the y-axis.

6. If $\vec{u}$ points SOUTHEAST and $\vec{v}$ points UP, where does $\vec{u} \times \vec{v}$ point?

- **a**. DOWN
- **b**. SOUTHWEST Correct
- **c**. WEST
- **d**. NORTHEAST
- **e**. NORTHWEST

Solution: Point the fingers of your right hand pointing SOUTHEAST with your palm facing UP. Your thumb points SOUTHWEST.

7. Find the intersection of the line $(x,y,z) = (2t, -1+2t, 2+2t)$ and the plane $3x - 2y + z = 8$. At this point $x + y + z =$

- **a**. -3
- **b**. -1
- **c**. 0
- **d**. 5
- **e**. 7 Correct

Solution: Plug the line into the plane and solve for t:
$3(2t) - 2(-1+2t) + (2+2t) = 8 \qquad 4t + 4 = 8 \qquad 4t = 4 \qquad t = 1$
Plug back into the line: $(x,y,z) = (2, -1+2, 2+2) = (2,1,4) \qquad x + y + z = 2 + 1 + 4 = 7$

8. Compute $\lim\limits_{h \to 0} \dfrac{(2x + 2h + 3y)^2 - (2x + 3y)^2}{h}$

- **a**. $2x + 3y$
- **b**. $4x + 6y$
- **c**. $6x + 9y$
- **d**. $8x + 12y$ Correct
- **e**. $12x + 18y$

Solution: $\dfrac{\partial}{\partial x}(2x + 3y)^2 = 2(2x + 3y)2 = 8x + 12y$

9. Find the plane tangent to the graph of $z = x^2 e^{2y}$ at $(3,0)$. The z-intercept is

a. -27
b. -18
c. -9 Correct
d. 9
e. 18

Solution:

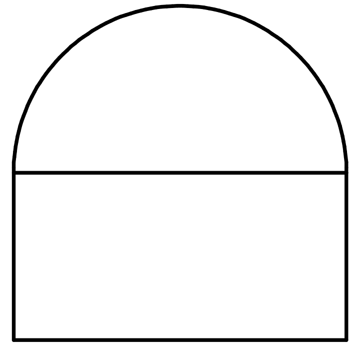
$$f(x,y) = x^2 e^{2y} \qquad f(3,0) = 9 \qquad z = f(3,0) + f_x(3,0)(x-3) + f_y(3,0)(y-0)$$
$$f_x(x,y) = 2xe^{2y} \qquad f_x(3,0) = 6 \qquad = 9 + 6(x-3) + 18(y)$$
$$f_y(x,y) = 2x^2 e^{2y} \qquad f_y(3,0) = 18 \qquad = 6x + 18y - 9 \qquad c = -9$$

10. If $S(3,2) = 5$ and $\frac{\partial S}{\partial x}(3,2) = -0.3$ and $\frac{\partial S}{\partial y}(3,2) = 0.4$, estimate $S(3.2,1.7)$.

a. 4.82 Correct
b. 4.9
c. 5.0
d. 5.1
e. 5.18

Solution: The linear approximation says:
$S(x,y) \approx S_{\tan}(x,y) = S(a,b) + S_x(a,b)(x-a) + S_y(a,b)(y-b)$
Here $(x,y) = (3.2,1.7)$ and $(a,b) = (3,2)$. So
$S(3.2,1.7) \approx S(3,2) + S_x(3,2)(.2) + S_y(3,2)(-.3) = 5 - 0.3(.2) + 0.4(-.3) = 4.82$

11. A semicircle sits on top of a rectangle of width $2r$ and height h. If the radius decreases from 3 cm to 2.97 cm while the height increases from 4 cm to 4.02 cm, use the linear approximation to determine whether the area increases or decreases and by how much.



a. increases by $0.09\pi - 0.12$
b. increases by $0.09\pi + 0.12$
c. increases by $0.09\pi + 0.36$
d. decreases by $0.09\pi + 0.36$
e. decreases by $0.09\pi + 0.12$ Correct

Solution: $A = 2rh + \frac{1}{2}\pi r^2$

$$\Delta A \approx dA = \frac{\partial A}{\partial r}dr + \frac{\partial A}{\partial h}dh = (2h + \pi r)dr + (2r)dh$$
$$= (2 \cdot 4 + \pi \cdot 3)(-.03) + (2 \cdot 3)(.02) = -0.09\pi - 0.12 < 0$$ decreases

12. The temperature in a room is $T = z^2(2x + 3y)$. Currently, a fly is at $\vec{r} = (4,3,2)$ and has velocity $\vec{v} = (3,2,1)$. What is the rate of change of the temperature as seen by the fly?

a. 16
b. 116 Correct
c. 64
d. 164
e. 264

Solution: $$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} + \frac{\partial T}{\partial z}\frac{dz}{dt} = (2z^2)v_1 + (3z^2)v_2 + (2z(2x + 3y))v_3$$
$$= (2 \cdot 2^2)3 + (3 \cdot 2^2)2 + (2 \cdot 2(2 \cdot 4 + 3 \cdot 3))1 = 116$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. $\left(16 \text{ points}\right)$ For the parametric curve $\vec{r}(t) = \left(\frac{2}{t}, 6t, 3t^3\right)$ compute each of the following:

a. velocity $\vec{v}$

Solution: $\vec{v} = \left(\frac{-2}{t^2}, 6, 9t^2\right)$

b. speed $|\vec{v}|$ HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{\frac{4}{t^4} + 36 + 81t^4} = \frac{2}{t^2} + 9t^2$

c. arc length $L = \int_{(2,6,3)}^{(1,12,24)} ds$

Solution: $L = \int_1^2 |\vec{v}|\,dt = \int_1^2 \left(\frac{2}{t^2} + 9t^2\right)dt = \left[\frac{-2}{t} + 3t^3\right]_1^2 = (-1+24) - (-2+3) = 22$

d. acceleration $\vec{a}$

Solution: $\vec{a} = \left(\frac{4}{t^3}, 0, 18t\right)$

e. unit binormal $\hat{B}$

Solution: $$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{-2}{t^2} & 6 & 9t^2 \\ \frac{4}{t^3} & 0 & 18t \end{vmatrix} = \hat{\imath}(108t) - \hat{\jmath}\left(\frac{-36}{t} - \frac{36}{t}\right) + \hat{k}\left(\frac{-24}{t^3}\right) = \left(108t, \frac{72}{t}, \frac{-24}{t^3}\right) = 12\left(9t, \frac{6}{t}, \frac{-2}{t^3}\right)$$

$$|\vec{v} \times \vec{a}| = 12\sqrt{81t^2 + \frac{36}{t^2} + \frac{4}{t^6}} = 12\left(9t + \frac{2}{t^3}\right) = \frac{12(9t^4+2)}{t^3}$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{t^3}{9t^4+2}\left(9t, \frac{6}{t}, \frac{-2}{t^3}\right) = \left(\frac{9t^4}{9t^4+2}, \frac{6t^2}{9t^4+2}, \frac{-2}{9t^4+2}\right)$$

f. tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\frac{2}{t^2} + 9t^2\right) = \frac{-4}{t^3} + 18t$

14. $\left(12 \text{ points}\right)$ A wire has the shape of the parametric curve $\vec{r}(t) = \left(\frac{2}{t}, 6t, 3t^3\right)$ between $(2,6,3)$ and $(1,12,24)$. Find the mass of the wire if the linear mass density is $\rho = \frac{1}{12}xyz$. Don't simplify the answer.

Solution: $\vec{v} = \left(\frac{-2}{t^2}, 6, 9t^2\right)$ $|\vec{v}| = \frac{2}{t^2} + 9t^2$ $\rho = \frac{1}{12}xyz = \frac{1}{12}\left(\frac{2}{t}\right)(6t)(3t^3) = 3t^3$

$$M = \int_{(2,6,3)}^{(1,12,12)} \rho\,ds = \int_1^2 \frac{1}{12}xyz|\vec{v}|\,dt = \int_1^2 3t^3\left(\frac{2}{t^2} + 9t^2\right)dt = \int_1^2 (6t + 27t^5)\,dt = \left[3t^2 + \frac{9t^6}{2}\right]_1^2$$
$$= \left(12 + \frac{9 \bullet 2^6}{2}\right) - \left(3 + \frac{9}{2}\right) = \frac{585}{2}$$

15. $\left(12 \text{ points}\right)$ A mass slides along a wire which has the shape of the parametric curve $\vec{r}(t) = \left(\frac{2}{t}, 6t, 3t^3\right)$ between $(2,6,3)$ and $(1,12,24)$ under the action of the force $\vec{F} = (z,y,x)$. Find the work done by the force.

Solution: $\vec{F} = (z,y,x) = \left(3t^3, 6t, \frac{2}{t}\right)$ $\vec{v} = \left(\frac{-2}{t^2}, 6, 9t^2\right)$

$$\vec{F} \bullet \vec{v} = 3t^3\frac{-2}{t^2} + 6t6 + \frac{2}{t}9t^2 = -6t + 36t + 18t = 48t$$

$$W = \int_{(2,6,3)}^{(1,12,12)} \vec{F} \bullet d\vec{s} = \int_1^2 \vec{F} \bullet \vec{v}\,dt = \int_1^2 48t\,dt = \left[24t^2\right]_1^2 = 24(4-1) = 72$$