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MATH 251 Exam 1B Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

1-12	/60
13	/16
14	/12
15	/12
Total	/100

Multiple Choice: (5 points each. No part credit.)

1. If $\vec{a} = (2,-1,2)$ and $\vec{b} = (1,2,5)$, then $\left|\vec{a} + 2\vec{b}\right| =$

a. 1

b. 3

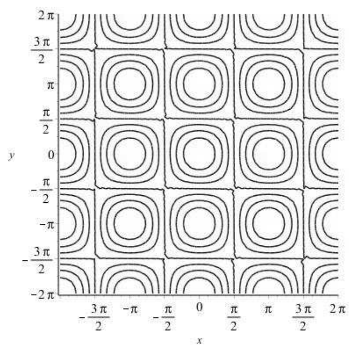
c. 5

d. 9

e. 13 Correct

Solution: $\left|\vec{a} + 2\vec{b}\right| = |(4,3,12)| = \sqrt{16+9+144} = 13$

2. The plot at the right is the contour plot of which function?
HINT: Where is the level set with value 0?



a. $\sin(x)\cos(y)$

b. $\sin(x)\sin(y)$

c. $\cos(x)\cos(y)$ Correct

d. $\cos(x)\sin(y)$

e. $\sin(xy)$

Solution: There are level sets at $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ but not $x = 0$ nor $y = 0$. So it must have a $\cos(x)$ and $\cos(y)$ factors but not $\sin(x)$ or $\sin(y)$ or $\sin(xy)$ factors.

3. Suppose $proj_{\vec{v}}\vec{u} = (3,1)$. Which of the following is **inconsistent** with this fact?

a. $proj_{\perp\vec{v}}\vec{u} = (2,-6)$
b. $proj_{\perp\vec{v}}\vec{u} = (-2,5)$ Correct
c. $\vec{u} = (4,-2)$
d. $\vec{v} = (6,2)$
e. $\vec{v} = (-3,-1)$

Solution: $proj_{\vec{v}}\vec{u}$ must be parallel to $\vec{v}$, and $(6,2)$ and $(-3,-1)$ are. $proj_{\vec{v}}\vec{u}$ must be perpendicular to $proj_{\perp\vec{v}}\vec{u}$, and $(2,-6)$ is but $(-2,5)$ is not because $(-2,5)\cdot(3,1) = -1 \neq 0$.

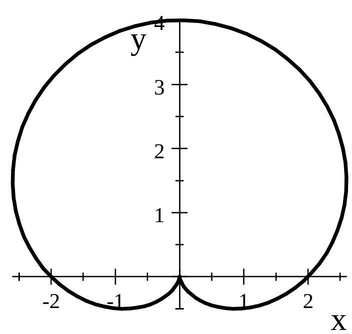
4. Find the asymptotes of the hyperbola $4(x-2)^2 - 9(y-3)^2 = 36$.

a. $y = 2 \pm \frac{3}{2}(x-3)$
b. $y = 2 \pm \frac{2}{3}(x-3)$
c. $y = 3 \pm \frac{3}{2}(x-2)$
d. $y = 3 \pm \frac{2}{3}(x-2)$ Correct
e. $y = -3 \pm \frac{2}{3}(x+2)$

Solution: In standard form the hyperbola is $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 1$. The asymptotes are the cross $\frac{(x-2)^2}{9} - \frac{(y-3)^2}{4} = 0$. We solve:

$$\frac{(y-3)^2}{4} = \frac{(x-2)^2}{9} \qquad (y-3)^2 = \frac{4}{9}(x-2)^2 \qquad y-3 = \pm\frac{2}{3}(x-2) \qquad y = 3 \pm \frac{2}{3}(x-2)$$

5. The plot at the right is the graph of which polar curve?



a. $r = 2 + 2\cos\theta$
b. $r = 2 - 2\cos\theta$
c. $r = 2 + 2\sin\theta$ Correct
d. $r = 2 - 2\sin\theta$

Solution: $r = 4$ at $\theta = \frac{\pi}{2}$ which is equation (c).

6. If $\vec{u}$ points SOUTHEAST and $\vec{v}$ points NORTH, where does $\vec{u} \times \vec{v}$ point?

- **a**. UP Correct
- **b**. DOWN
- **c**. SOUTHWEST
- **d**. WEST
- **e**. NORTHEAST

Solution: Point the fingers of your right hand pointing SOUTHEAST with your palm facing NORTH. Your thumb points UP.

7. Find the plane through the points $A = (2,3,4)$, $B = (1,3,5)$ and $C = (2,1,5)$. Its z-intercept is:

- **a**. 0
- **b**. 5
- **c**. 10
- **d**. 15
- **e**. $\frac{15}{2}$ Correct

Solution: $\vec{u} = \overrightarrow{AB} = (-1,0,1)$ $\quad \vec{v} = \overrightarrow{AC} = (0,-2,1)$ $\quad \vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & -2 & 1 \end{vmatrix} = (2,1,2)$

$\vec{N} \cdot X = \vec{N} \cdot A \qquad 2x + y + 2z = 2 \cdot 2 + 3 + 2 \cdot 4 = 15 \qquad$ z-intercept $= \frac{15}{2}$

8. Compute $\lim_{h \to 0} \frac{(2x+3y+3h)^2 - (2x+3y)^2}{h}$

- **a**. $2x + 3y$
- **b**. $4x + 6y$
- **c**. $6x + 9y$
- **d**. $8x + 12y$
- **e**. $12x + 18y$ Correct

Solution: $\frac{\partial}{\partial y}(2x+3y)^2 = 2(2x+3y)3 = 12x + 18y$

9. Find the plane tangent to the graph of $z = x^3 e^{2y}$ at $(2,1)$. The z-intercept is

a. $-32e^2$ Correct

b. $-8e^2$

c. 0

d. $8e^2$

e. $32e^2$

Solution:

$$f(x,y) = x^3 e^{2y} \qquad f(2,1) = 8e^2 \qquad z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$f_x(x,y) = 3x^2 e^{2y} \qquad f_x(2,1) = 12e^2 \qquad = 8e^2 + 12e^2(x-2) + 16e^2(y-1)$$

$$f_y(x,y) = 2x^3 e^{2y} \qquad f_y(2,1) = 16e^2 \qquad = 12e^2 x + 16e^2 y - 32e^2 \qquad c = -32e^2$$

10. If $T(3,2) = 4$ and $\frac{\partial T}{\partial x}(3,2) = -0.4$ and $\frac{\partial T}{\partial y}(3,2) = 0.2$, estimate $T(2.8,2.3)$.

a. 3.7

b. 3.8

c. 3.86

d. 3.9

e. 4.14 Correct

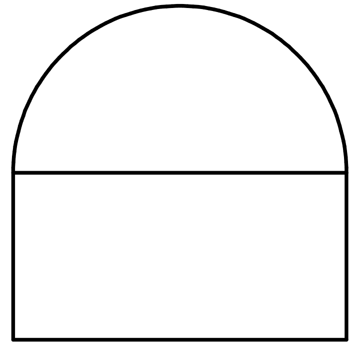
Solution: The linear approximation says:

$T(x,y) \approx T_{\tan}(x,y) = T(a,b) + T_x(a,b)(x-a) + T_y(a,b)(y-b)$

Here $(x,y) = (2.8,2.3)$ and $(a,b) = (3,2)$. So

$T(2.8,2.3) \approx T(3,2) + T_x(3,2)(-.2) + T_y(3,2)(.3) = 4 - 0.4(-.2) + 0.2(.3) = 4.14$

11. A semicircle sits on top of a rectangle of width $2r$ and height h. If the radius increases from 3 cm to 3.03 cm while the height decreases from 4 cm to 3.98 cm, use the linear approximation to determine whether the area increases or decreases and by how much.



a. increases by $0.09\pi - 0.12$
b. increases by $0.09\pi + 0.12$ Correct
c. increases by $0.09\pi + 0.36$
d. decreases by $0.09\pi + 0.36$
e. decreases by $0.09\pi + 0.12$

Solution: $A = 2rh + \frac{1}{2}\pi r^2$

$$\Delta A \approx dA = \frac{\partial A}{\partial r}dr + \frac{\partial A}{\partial h}dh = (2h + \pi r)dr + (2r)dh$$
$$= (2 \cdot 4 + \pi \cdot 3)(.03) + (2 \cdot 3)(-.02) = 0.09\pi + 0.12 > 0$$ increases

12. The brightness of a candle at the origin seen from the point (x,y,z) is $B = \frac{1}{x^2 + y^2 + z^2}$. A moth is at $\vec{r} = (-1,2,2)$ and has velocity $\vec{v} = (3,2,1)$. What is the rate of change of the brightness as seen by the moth?

a. $-\frac{2}{3}$
b. $-\frac{2}{27}$ Correct
c. $-\frac{2}{81}$
d. $-\frac{3}{4}$
e. $\frac{15}{16}$

Solution:
$$\frac{dB}{dt} = \frac{\partial B}{\partial x}\frac{dx}{dt} + \frac{\partial B}{\partial y}\frac{dy}{dt} + \frac{\partial B}{\partial z}\frac{dz}{dt} = \frac{-2x}{(x^2+y^2+z^2)^2}v_1 + \frac{-2y}{(x^2+y^2+z^2)^2}v_2 + \frac{-2z}{(x^2+y^2+z^2)^2}v_3$$
$$= \frac{-2(-1)}{(9)^2}3 + \frac{-2(2)}{(9)^2}2 + \frac{-2(2)}{(9)^2}1 = -\frac{2}{27}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

13. (16 points) For the parametric curve $\vec{r}(t) = \left(\frac{2}{3}t, t^2, t^3\right)$ compute each of the following:

a. velocity $\vec{v}$

Solution: $\vec{v} = \left(\frac{2}{3}, 2t, 3t^2\right)$

b. speed $|\vec{v}|$ HINT: The quantity inside the square root is a perfect square.

Solution: $|\vec{v}| = \sqrt{\frac{4}{9} + 4t^2 + 9t^4} = \frac{2}{3} + 3t^2$

c. arc length $L = \int_{(0,0,0)}^{(2,9,27)} ds$

Solution: $L = \int_0^3 |\vec{v}|\,dt = \int_0^3 \left(\frac{2}{3} + 3t^2\right) dt = \left[\frac{2}{3}t + t^3\right]_0^3 = 2 + 27 = 29$

d. acceleration $\vec{a}$

Solution: $\vec{a} = (0, 2, 6t)$

e. unit binormal $\hat{B}$

Solution:
$$\vec{v} \times \vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{2}{3} & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \hat{\imath}(12t^2 - 6t^2) - \hat{\jmath}(4t) + \hat{k}\left(\frac{4}{3}\right) = \left(6t^2, -4t, \frac{4}{3}\right)$$

$$|\vec{v} \times \vec{a}| = \sqrt{36t^4 + 16t^2 + \frac{16}{9}} = 6t^2 + \frac{4}{3} = \frac{2(9t^2 + 2)}{3}$$

$$\hat{B} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|} = \frac{3}{2(9t^2 + 2)}\left(6t^2, -4t, \frac{4}{3}\right) = \left(\frac{9t^2}{9t^2 + 2}, \frac{-6t}{9t^2 + 2}, \frac{2}{9t^2 + 2}\right)$$

f. tangential acceleration a_T

Solution: $a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(\frac{2}{3} + 3t^2\right) = 6t$

14. (12 points) A wire has the shape of the parametric curve $\vec{r}(t) = \left(\frac{2}{3}t, t^2, t^3\right)$ between $(0,0,0)$ and $(2,9,27)$. Find the mass of the wire if the linear mass density is $\rho = yz$. Don't simplify the answer.

Solution: $\vec{v} = \left(\frac{2}{3}, 2t, 3t^2\right)$ $|\vec{v}| = \frac{2}{3} + 3t^2$ $\rho = yz = (t^2)(t^3) = t^5$

$$M = \int_{(0,0,0)}^{(2,9,27)} \rho\,ds = \int_0^3 yz|\vec{v}|\,dt = \int_0^3 t^5\left(\frac{2}{3} + 3t^2\right) dt = \int_0^3 \left(\frac{2}{3}t^5 + 3t^7\right) dt = \left[\frac{t^6}{9} + \frac{3t^8}{8}\right]_0^3$$
$$= \left(3^4 + \frac{3^9}{8}\right) = \frac{20\,331}{8}$$

15. (12 points) A mass slides along a wire which has the shape of the parametric curve $\vec{r}(t) = \left(\frac{2}{3}t, t^2, t^3\right)$ between $(0,0,0)$ and $(2,9,27)$ under the action of the force $\vec{F} = (3z, 2y, x)$. Find the work done by the force.

Solution: $\vec{F} = (3z, 2y, x) = \left(3t^3, 2t^2, \frac{2}{3}t\right)$ $\vec{v} = \left(\frac{2}{3}, 2t, 3t^2\right)$

$$\vec{F} \cdot \vec{v} = 3t^3\frac{2}{3} + 2t^2 2t + \frac{2}{3}t3t^2 = 8t^3$$

$$W = \int_{(0,0,0)}^{(2,9,27)} \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F} \cdot \vec{v}\,dt = \int_0^3 8t^3\,dt = \left[2t^4\right]_0^3 = 2 \cdot 81 = 162$$