Name________________________

1-9	/45
10	/30
11	/15
12	/15
Total	/105

MATH 251 Exam 2B Fall 2015

Sections 511/512 (circle one) Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the volume below the surface $z = 3x + y$ above the region in the xy-plane between $y = x^2$ and $y = 2x$.

a. -32

b. 32

c. $\dfrac{32}{15}$

d. $\dfrac{92}{15}$ Correct

e. $\dfrac{116}{15}$

Solution: Find the intersections: $x^2 = 2x \quad \Rightarrow \quad x = 0,2$

$$V = \int_0^2 \int_{x^2}^{2x} (3x + y)\,dy\,dx = \int_0^2 \left[3xy + \frac{y^2}{2} \right]_{y=x^2}^{2x} dx = \int_0^2 (6x^2 + 2x^2) - \left(3x^3 + \frac{x^4}{2}\right) dx$$

$$= \left[8\frac{x^3}{3} - 3\frac{x^4}{4} - \frac{x^5}{10} \right]_0^2 = \left(8 \cdot \frac{8}{3} - 3 \cdot 4 - \frac{16}{5} \right) = \frac{92}{15}$$

2. The temperature on a circular hot plate with radius 2 is $T = x^2 + 4$.
Find the average temperature.

a. 10π

b. 20π

c. $4\pi^2 + 5\pi$

d. $\dfrac{4\pi + 5}{4}$

e. 5 Correct

Solution: $A = \int_0^{2\pi} \int_0^2 1\,r\,dr\,d\theta = [\pi r^2]_0^2 = 4\pi$

$T = x^2 + 4 = r^2 \cos^2\theta + 4$

$$\iint T\,dA = \int_0^{2\pi} \int_0^2 (r^2 \cos^2\theta + 4)\,r\,dr\,d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \cos^2\theta + 4\frac{r^2}{2} \right]_{r=0}^2 d\theta$$

$$= \int_0^{2\pi} \left(4\frac{1 + \cos(2\theta)}{2} + 8 \right) d\theta = \left[2\left(\theta + \frac{\sin(2\theta)}{2}\right) + 8\theta \right]_0^{2\pi} = 4\pi + 16\pi = 20\pi$$

$$T_{ave} = \frac{1}{A} \iint T\,dA = \frac{20\pi}{4\pi} = 5$$

3. Find the centroid of the region above $y = 3x^2$ below $y = 12$.

a. $\left(0, \frac{27}{5}\right)$

b. $\left(0, \frac{36}{5}\right)$ Correct

c. $\left(0, \frac{54}{5}\right)$

d. $\left(0, \frac{1944}{5}\right)$

e. $\left(0, \frac{3888}{5}\right)$

Solution: $3x^2 = 12 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

$$A = \iint 1\,dA = \int_{-2}^{2}\int_{3x^2}^{12} 1\,dy\,dx = \int_{-2}^{2}(12 - 3x^2)\,dx = [12x - x^3]_{-2}^{2} = 2(24 - 8) = 32$$

By symmetry, $\bar{x} = 0$.

$$A_y = \iint y\,dA = \int_{-2}^{2}\int_{3x^2}^{12} y\,dy\,dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_{3x^2}^{12} dx = \frac{1}{2}\int_{-2}^{2}(12^2 - 9x^4)\,dx = \frac{1}{2}\left[12^2x - 9\frac{x^5}{5}\right]_{-2}^{2}$$

$$= \left(12^2 \cdot 2 - 9\frac{2^5}{5}\right) = 2^5\left(9 - \frac{9}{5}\right) = \frac{36}{5}2^5$$

$$\bar{y} = \frac{A_y}{A} = \frac{36 \cdot 2^5}{5 \cdot 32} = \frac{36}{5}$$

4. Find all critical points of the function $f(x,y) = 9x^2 + 4y^2 + \dfrac{432}{xy}$. Select from:

$A = (2,3) \quad B = (-2,3) \quad C = (2,-3) \quad D = (-2,-3)$

$E = (3,2) \quad F = (-3,2) \quad G = (3,-2) \quad H = (-3,-2)$

Note $432 = 2^4 3^3$

a. A,B,C,D

b. E,F,G,H

c. A,D Correct

d. B,C

e. E,H

Solution:

$$f_x = \frac{d}{dx}\left(9x^2 + 4y^2 + \frac{432}{xy}\right) = \frac{18}{x^2y}(x^3y - 24) \qquad x^3y = 24$$

$$f_y = \frac{d}{dy}\left(9x^2 + 4y^2 + \frac{432}{xy}\right) = \frac{8}{xy^2}(xy^3 - 54) \qquad xy^3 = 54$$

Multiply: $x^4y^4 = 24 \cdot 54 = 2^4 3^4 \quad \Rightarrow \quad xy = \pm 6$

Divide: $\dfrac{x^2}{y^2} = \dfrac{24}{54} = \dfrac{4}{9} \quad \Rightarrow \quad \dfrac{x}{y} = \pm\dfrac{2}{3}$

Multiply results: $xy\dfrac{x}{y} = x^2 = \pm 6 \cdot \dfrac{2}{3} = \pm 4$ Must have $x^2 = 4 \quad x = \pm 2$

Divide results: $xy\dfrac{y}{x} = y^2 = \pm 6 \cdot \dfrac{3}{2} = \pm 9$ Must have $y^2 = 9 \quad y = \pm 3$

Looking at $x^3y = 24$, x and y must be both positive or both negative. So $(2,3), (-2,-3)$

5. Select all of the following statements which are consistent with this countour plot?

A. There is a local maximum at $(1,0)$.

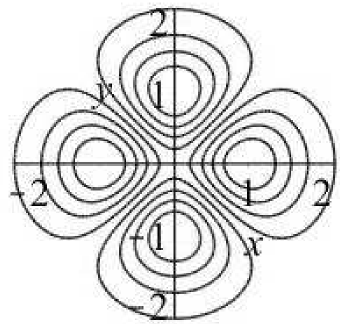
B. There is a local minimum at $(1,0)$.

C. There is a saddle point at $(1,0)$.

D. There is a local maximum at $(0,0)$.

E. There is a local minimum at $(0,0)$.

F. There is a saddle point at $(0,0)$.



a. A,B,D,E

b. C,F

c. A,B,F   Correct

d. C,D,E

Solution: A,B but not D,E because there should be circles around a local maximum or minimum.

6. The function $f = \frac{4}{x} + \frac{2}{y} + xy$ has a critical point at $(x,y) = (2,1)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum   Correct

b. Local Maximum

c. Inflection Point

d. Saddle Point

e. Test Fails

Solution: $f_x = -\frac{4}{x^2} + y \qquad f_y = -\frac{2}{y^2} + x$

$f_{xx} = \frac{8}{x^3} \qquad f_{yy} = \frac{4}{y^3} \qquad f_{xy} = 1$

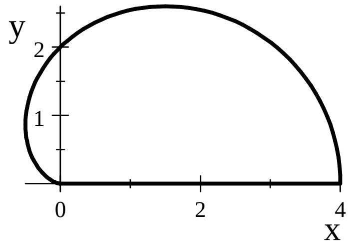
$f_{xx}(2,1) = 1 \qquad f_{yy}(2,1) = 4 \qquad f_{xy}(2,1) = 1 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3$

Local Minimum

7. Find the mass of the region inside the upper half of the cardioid

$r = 2 + 2\cos\theta$

if the surface density is $\delta = y$.



a. $\dfrac{16}{3}$

b. $\dfrac{32}{3}$ Correct

c. 4

d. $\dfrac{16}{9}$

e. $\dfrac{32}{9}$

Solution: The density is $\delta = y = r\sin\theta$.

$$M = \iint \delta\, dA = \int_0^\pi \int_0^{2+2\cos\theta} r\sin\theta\, r\, dr\, d\theta = \int_0^\pi \left[\frac{r^3}{3} \right]_0^{2+2\cos\theta} \sin\theta\, d\theta = \frac{1}{3} \int_0^\pi (2+2\cos\theta)^3 \sin\theta\, d\theta$$

Let $u = 2 + 2\cos\theta \qquad du = -2\sin\theta\, d\theta \qquad \dfrac{-1}{2} du = \sin\theta\, d\theta$

$$M = \frac{-1}{6} \int_4^0 u^3\, du = \frac{-1}{6} \left[\frac{u^4}{4} \right]_4^0 = \frac{-1}{6}(0 - 64) = \frac{32}{3}$$

8. Find the x-component of the center of mass of the region inside the upper half of the cardioid $r = 2 + 2\cos\theta$ if the surface density is $\delta = y$.

a. $\dfrac{4}{5}$

b. $\dfrac{6}{5}$

c. $\dfrac{8}{5}$ Correct

d. $\dfrac{32}{5}$

e. $\dfrac{256}{15}$

Solution: The density is $\delta = y = r\sin\theta$ and $x = r\cos\theta$.

$$M_x = \iint x\,\delta\, dA = \int_0^\pi \int_0^{2+2\cos\theta} r\cos\theta\, r\sin\theta\, r\, dr\, d\theta = \int_0^\pi \left[\frac{r^4}{4} \right]_0^{2+2\cos\theta} \cos\theta \sin\theta\, d\theta$$

$$= \frac{1}{4} \int_0^\pi (2+2\cos\theta)^4 \cos\theta \sin\theta\, d\theta$$

Let $u = 2 + 2\cos\theta \qquad du = -2\sin\theta\, d\theta \qquad \dfrac{-1}{2} du = \sin\theta\, d\theta \qquad \cos\theta = \dfrac{1}{2}(u-2)$

$$M_x = \frac{-1}{16} \int_4^0 u^4 (u-2)\, du = \frac{-1}{16} \left[\frac{u^6}{6} - 2\frac{u^5}{5} \right]_4^0 = \frac{256}{15} \qquad \bar{x} = \frac{M_x}{M} = \frac{256}{15}\,\frac{3}{32} = \frac{8}{5}$$

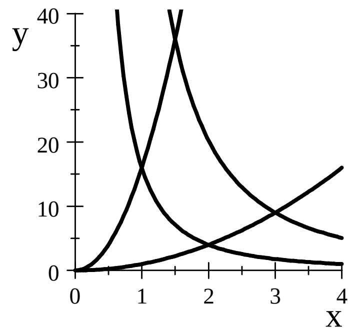
9. Compute $\iint x\,dA$ over the "diamond" shaped region in the first quadrant bounded by

$$x^2y = 16 \qquad x^2y = 81 \qquad y = x^2 \qquad y = 16x^2$$

HINTS: Use the coordinates

$$x = \frac{u}{v} \qquad y = u^2v^2$$

Find the boundaries and Jacobian.



a. $19\ln 2$

b. $80\ln 2$

c. $15\ln 3$

d. $65\ln 2$ Correct

e. $65\ln 3$

Solution: Let $(x,y) = (\frac{u}{v}, u^2v^2)$

Boundaries: $x^2y = (\frac{u}{v})^2 u^2v^2 = u^4 = 16, 81 \qquad \Rightarrow \qquad u = 2, 3$

$$\frac{y}{x^2} = u^2v^2\frac{v^2}{u^2} = v^4 = 1, 16 \qquad \Rightarrow \qquad v = 1, 2$$

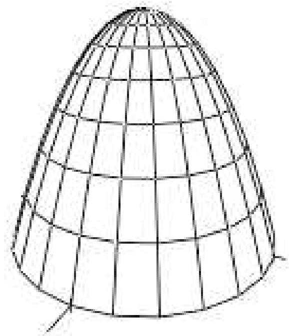
Jacobian: $$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{1}{v} & 2uv^2 \\ -\frac{u}{v^2} & 2u^2v \end{matrix}\right\| = |2u^2 - -2u^2| = 4u^2$$

Evaluate the integral:

$$\iint x\,dA = \int\int x\,J\,du\,dv = \int_1^2\int_2^3 \frac{u}{v}\,4u^2\,du\,dv = [u^4]_2^3[\ln v]_1^2 = (81-16)\ln 2 = 65\ln 2$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (30 points) Consider the piece of the paraboloid surface $z = 12 - 3x^2 - 3y^2$ above the xy-plane.



- Find the mass of the paraboloid if the surface mass density is $\delta = z + 3x^2 + 3y^2$.
- Find the flux of the electric field $\vec{E} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, 0 \right)$ down into the paraboloid.

Parametrize the surface as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, 12 - 3r^2)$ and follow these steps:

a. Find the coordinate tangent vectors:

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & -6r \\ (-r\sin\theta & r\cos\theta & 0 \end{array} \right|$$

b. Find the normal vector and check its orientation.

$\vec{N} = i(6r^2\cos\theta) - j(-6r^2\sin\theta) + k(r\cos^2\theta - -r\sin^2\theta) = (6^2\cos\theta, 6r^2\sin\theta, r)$
This is up and out. We need down and in. So reverse the normal.

$\vec{N} = (-6r^2\cos\theta, -6r^2\sin\theta, -r)$

c. Find the length of the normal vector.

$\left|\vec{N}\right| = \sqrt{36r^4\cos^2\theta + 36r^4\sin^2\theta + r^2} = \sqrt{36r^4 + r^2} = r\sqrt{36r^2+1}$

d. Evaluate the density $\delta = z + 3x^2 + 3y^2$ on the paraboloid.

$$\delta\left(\vec{R}(r,\theta)\right) = (12 - 3r^2) + 3r^2 = 12$$

e. Compute the mass.

Find the limit on r: $\quad z = 12 - 3r^2 = 0 \quad \Rightarrow \quad r = 2$

$$M = \iint \delta\, dS = \int_0^{2\pi}\int_0^2 12r\sqrt{36r^2+1}\; dr\, d\theta = 2\pi \cdot 12\left[\frac{2}{3 \cdot 72}(36r^2+1)^{3/2}\right]_0^2$$
$$= \frac{2\pi}{9}(145^{3/2} - 1)$$

f. Evaluate the electric field $\vec{E} = \left(\dfrac{x}{x^2+y^2}, \dfrac{y}{x^2+y^2}, 0\right)$ on the paraboloid.

$$\vec{E}\left(\vec{R}(r,\theta)\right) = \left(\frac{r\cos\theta}{r^2}, \frac{r\sin\theta}{r^2}, 0\right) = \left(\frac{\cos\theta}{r}, \frac{\sin\theta}{r}, 0\right)$$

g. Compute the flux.

$$\iint \vec{E} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 \vec{E} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 \left(-\frac{\cos\theta}{r}6r^2\cos\theta - \frac{\sin\theta}{r}6r^2\sin\theta\right) dr\, d\theta$$
$$= 2\pi\int_0^2 (-6r)\, dr = -12\pi\left[\frac{r^2}{2}\right]_0^2 = -24\pi$$

11. (15 points) Find the point in the first octant on the surface, $z(x+y) = 2\sqrt{2}$, closest to the origin.

Solution: Minimize $f = D^2 = x^2 + y^2 + z^2$ subject to the constraint $g = z(x+y) = 2\sqrt{2}$.
Use Lagrange Multipliers

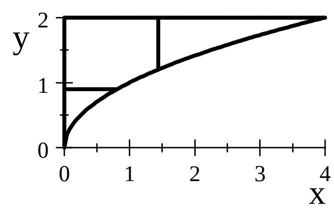
$$\vec{\nabla} f = (2x, 2y, 2z) \qquad \vec{\nabla} g = (z, z, x+y) \qquad \vec{\nabla} f = \lambda \vec{\nabla} g \qquad 2x = \lambda z \qquad 2y = \lambda z \qquad 2z = \lambda(x+y)$$

$$\lambda = \frac{2x}{z} = \frac{2y}{z} = \frac{2z}{x+y} \quad \Rightarrow \quad x = y, \quad \frac{2x}{z} = \frac{2z}{2x} \quad \Rightarrow \quad 2x^2 = z^2 \quad \Rightarrow \quad z = \sqrt{2}\,x$$

$g = z(x+y) = \sqrt{2}\,x(2x) = 2\sqrt{2}\,x^2 = 2\sqrt{2}$ So $x = 1$ $y = 1$ $z = \sqrt{2}$

12. (15 points) Draw the region of integration and compute $\int_0^4 \int_{\sqrt{x}}^2 \sqrt{y^3+1}\,dy\,dx$

Solution: To reverse the order of integration plot the region $0 \le x \le 4$, $\sqrt{x} \le y \le 2$.
Include a vertical line to indicate the y limits.
Add a horizontal line to indicate the new x limits.
Write the new integral and compute it.



$$\int_0^2 \int_0^{y^2} \sqrt{y^3+1}\,dx\,dy = \int_0^2 \sqrt{y^3+1}\,[x]_0^{y^2}\,dy = \int_0^2 \sqrt{y^3+1}\,y^2\,dy = \frac{2}{9}(y^3+1)^{3/2}\Big|_0^2 = \frac{2}{9}(9^{3/2}-1) = \frac{52}{9}$$