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MATH 251 Final Exam Fall 2015

Sections 512 Version A Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. A triangle has vertices $A = (2,2,1)$, $B = (3,4,2)$ and $C = (2,5,4)$. Find the angle at A.

a. $30°$ Correct Choice

b. $45°$

c. $60°$

d. $120°$

e. $150°$

Solution: $\overrightarrow{AB} = (1,2,1)$ $\overrightarrow{AC} = (0,3,3)$

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|} = \frac{9}{\sqrt{6}\,3\sqrt{2}} = \frac{\sqrt{3}}{2} \qquad \theta = 30°$$

2. Find the tangent plane to the graph of $z = x^2y + y^3x^3$ at $(x,y) = (2,1)$.
Where does it cross the z-axis?

a. -72

b. -48 Correct Choice

c. -12

d. 12

e. 48

Solution:

$$f(x,y) = x^2y + y^3x^3 \qquad f(2,1) = 12$$
$$f_x(x,y) = 2xy + 3y^3x^2 \qquad f_x(2,1) = 16$$
$$f_y(x,y) = x^2 + 3y^2x^3 \qquad f_y(2,1) = 28$$

$$z = f_{\tan}(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 12 + 16(x-2) + 28(y-1)$$

z-intercept $= f_{\tan}(0,0) = 12 + 16(-2) + 28(-1) = -48$

3. Find the tangent plane to the graph of hyperbolic paraboloid

$$9(x-3)^2 - 4(y-1)^2 - 4(z-2)^2 = 1$$

at the point $(2,2,3)$. Where does it cross the z-axis?

a. -19

b. $-\frac{19}{2}$

c. 0

d. $\frac{19}{2}$ Correct Choice

e. 19

Solution: $P = (2,2,3)$ $F = 9(x-3)^2 - 4(y-1)^2 - 4(z-2)^2$

$\vec{\nabla}F = (18(x-3), -8(y-1), -8(z-2))$ $\vec{N} = \vec{\nabla}F\big|_P = (-18,-8,-8)$

$\vec{N} \cdot X = \vec{N} \cdot P$ $-18x - 8y - 8z = -18 \cdot 2 - 8 \cdot 2 - 8 \cdot 3 = -76$

z-intercept at $x = y = 0$: $-8z = -76$ $z = \frac{76}{8} = \frac{19}{2}$

4. Sketch the region of integration for the integral $\int_0^{\pi/4}\int_{\tan(y)}^1 \frac{1}{\arctan(x)}\,dx\,dy$ in problem (12), then select its value here:

a. $\frac{1}{2}$

b. $\frac{\pi}{4}$

c. $\frac{\pi}{3}$

d. $\frac{\pi}{2}$

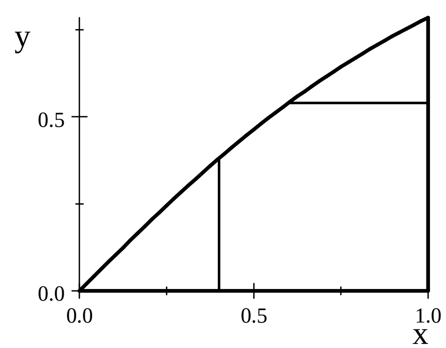
e. 1 Correct Choice

Solution: Reverse the order of integration:

$$\int_0^{\pi/4}\int_{\tan(y)}^1 \frac{1}{\arctan(x)}\,dx\,dy = \int_0^1\int_0^{\arctan(x)} \frac{1}{\arctan(x)}\,dy\,dx$$

$$= \int_0^1 \left[\frac{y}{\arctan(x)}\right]_{y=0}^{\arctan(x)} dx$$

$$= \int_0^1 \frac{\arctan(x)}{\arctan(x)}\,dx = [x]_0^1 = 1$$



5. Ham Duet is flying the Millenium Eagle through the galaxy on the path $\vec{r}(t) = (t, t^2, t^3)$. At $t = 2$ hours, he releases a trash pod which travels along the tangent line to the path of the Eagle with constant velocity equal to the velocity of the Eagle at time of release. Where is the trash pod 1 hours after release?

a. $(3,8,20)$ Correct Choice

b. $(3,9,27)$

c. $(2,4,8)$

d. $(1,2,12)$

e. $(5,10,43)$

Solution: $P = \vec{r}(2) = (2,4,8)$ $\quad \vec{v} = (1,2t,3t^2)$ $\quad \vec{v}(2) = (1,4,12)$

The path of the trash pod is $X = P + t\vec{v} = (2,4,8) + t(1,4,12) = (2+t, 4+4t, 8+12t)$

1 hour after release, the trash pod is at $X(1) = (2+1, 4+4, 8+12) = (3,8,20)$

6. Consider the surface of the cone given in cylindrical coordinates by $z = 4 - r$ above the xy-plane. It may be parametrized by

$$R(r,\theta) = (r\cos\theta, r\sin\theta, 4 - r).$$

Its temperature is $T = z$. Find its average temperature.

a. $\dfrac{32\sqrt{2}\,\pi}{3}$

b. $16\sqrt{2}\,\pi$

c. $\dfrac{2}{3}$

d. $\dfrac{4}{3}$ Correct Choice

e. $\dfrac{8}{3}$

Solution: We first find the normal and its length:

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta & -1) \\ (-r\sin\theta & r\cos\theta & 0\ ) \end{array} \right|$$

$$\vec{N} = \vec{e}_\phi \times \vec{e}_\theta = \hat{\imath}(--r\cos\theta) - \hat{\jmath}(-r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = \langle r\cos\theta, r\sin\theta, r\rangle$$

$$\left|\vec{N}\right| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2}\,r$$

The surface area is

$$S = \int_0^{2\pi}\int_0^4 \sqrt{2}\,r\,dr\,d\theta = 2\pi\sqrt{2}\left[\frac{r^2}{2}\right]_0^4 = 16\sqrt{2}\,\pi$$

The integral of the temperature, $T = z = 4 - r$, is

$$\iint T\,dS = \int_0^{2\pi}\int_0^4 (4-r)\sqrt{2}\,r\,dr\,d\theta = 2\pi\sqrt{2}\int_0^4 (4r - r^2)\,dr$$

$$= 2\pi\sqrt{2}\left[2r^2 - \frac{r^3}{3}\right]_0^4 = 2\pi\sqrt{2}\left(32 - \frac{64}{3}\right) = \frac{64\sqrt{2}\,\pi}{3}$$

So the average temperature is

$$T_{\text{ave}} = \frac{1}{A}\iint T\,dS = \frac{1}{16\sqrt{2}\,\pi}\,\frac{64\sqrt{2}\,\pi}{3} = \frac{4}{3}$$

7. Death star is basically a spherical shell with a hole cut out of one end, which we will take as centered at the south pole. In spherical coordinates, it fills the region between $1 \le \rho \le 4$ and $0 \le \phi \le \frac{2\pi}{3}$. Find the volume.

a. $V = 21\pi$

b. $V = 42\pi$

c. $V = 63\pi$ Correct Choice

d. $V = 105\pi$

e. $V = 126\pi$

Solution: The volume is

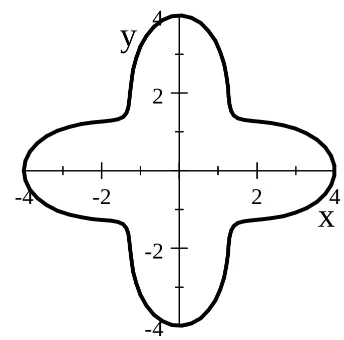
$$V = \int_0^{2\pi} \int_0^{2\pi/3} \int_1^4 1\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi[-\cos\phi]_0^{2\pi/3} \left[\frac{\rho^3}{3} \right]_1^4$$
$$= 2\pi\left[-\cos\frac{2\pi}{3} - -\cos 0 \right]\left[\frac{4^3 - 1}{3} \right] = 2\pi\left(- -\frac{1}{2} - -1 \right)21 = 63\pi$$

8. Compute $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s}$ for $f = xe^y$ counterclockwise around the polar curve

$r = 3 + \cos(4\theta)$

shown at the right.

Hint: Use a Theorem.



a. $4e^3$

b. $3e^4$

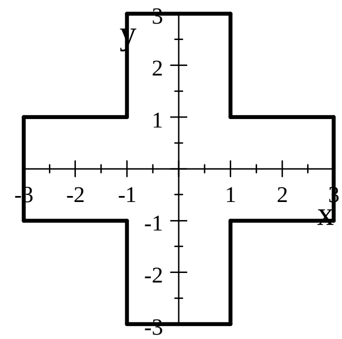
c. $4e^4 - 3e^3$

d. $3e^3 - 4e^4$

e. 0 Correct Choice

Solution: By the FTCC, $\oint_{\partial R} \vec{\nabla} f \cdot d\vec{s} = f(B) - f(A) = 0$ because $B = A$ no matter what point you start at.

9. Compute $\oint\limits_{\partial P} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (4x^3 - 2y, x + 3y^4)$ counterclockwise around the complete boundary of the plus sign shown at the right. Hint: Use a Theorem.



a. 120

b. 60 Correct Choice

c. 20

d. -20

e. -60

Solution: $P = 4x^3 - 2y \qquad Q = x + 3y^4 \qquad \partial_x Q - \partial_y P = 1 - -2 = 3$

By Green's Theorem,

$$\oint\limits_{\partial R} \vec{F} \cdot d\vec{s} = \iint\limits_{R} (\partial_x Q - \partial_y P)\,dx\,dy = \iint\limits_{R} 3\,dx\,dy = 3\left(\text{area}\right) = 3(20) = 60$$

10. Compute $\iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ over the complete surface of the hemisphere $0 \le z \le \sqrt{9 - x^2 - y^2}$ oriented outward,for $\vec{F} = \left(\frac{2}{3}x^3 z, \frac{2}{3}y^3 z, \frac{1}{2}z^4\right)$

Hint: Use a Theorem.

a. 27π

b. 54π

c. 81π

d. 162π

e. 243π Correct Choice

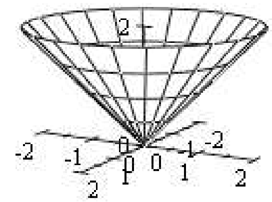
Solution: By Gauss' Theorem $\iint\limits_{\partial H} \vec{F} \cdot d\vec{S} = \iiint\limits_{H} \vec{\nabla} \cdot \vec{F}\,dV$. The divergence is $\vec{\nabla} \cdot \vec{F} = 2x^2 z + 2y^2 z + 2z^3 = 2z(x^2 + y^2 + z^2) = 2\rho^3 \cos\phi$ and the volume element is $dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$. So the integral is:

$$\iiint\limits_{H} \vec{\nabla} \cdot \vec{F}\,dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 2\rho^3 \cos\phi\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} \left[2\frac{\rho^6}{6}\right]_0^3 = 3^5\pi = 243\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (30 points) Verify Stokes' Theorem $\iint_C \vec{\nabla}\times\vec{F}\bullet d\vec{S} = \oint_{\partial C} \vec{F}\bullet d\vec{s}$



for the vector field $\vec{F} = (yz^2, -xz^2, z^3)$ and the surface of the cone $z = \sqrt{x^2+y^2}$ for $z \le 2$, oriented down and out.

Be careful with orientations. Use the following steps:

Left Hand Side:

The cone, C, may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

a. Compute the tangent vectors:

$$\vec{e}_r = (\cos\theta, \quad \sin\theta, \quad 1)$$
$$\vec{e}_\theta = (-r\sin\theta, \quad r\cos\theta, \quad 0)$$

b. Compute the normal vector:

$$\vec{N} = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(- -r\sin\theta) + \hat{k}(r\cos^2\theta - -r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

This is oriented up and in. Need down and out. Reverse: $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

c. Compute the curl of the vector field $\vec{F} = (yz^2, -xz^2, z^3)$:

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ yz^2 & -xz^2 & z^3 \end{vmatrix} = \hat{\imath}(0 - -2xz) - \hat{\jmath}(0 - 2yz) + \hat{k}(-z^2 - z^2) = (2xz, 2yz, -2z^2)$$

d. Evaluate the curl of $\vec{F}$ on the cone:

$$\vec{\nabla}\times\vec{F}\Big|_{\vec{R}(r,\theta)} = (2r^2\cos\theta, 2r^2\sin\theta, -2r^2)$$

e. Compute the dot product of the curl of $\vec{F}$ and the normal $\vec{N}$.

$$\vec{\nabla}\times\vec{F}\bullet\vec{N} = 2r^3\cos^2\theta + 2r^3\sin^2\theta + 2r^3 = 4r^3$$

f. Compute the left hand side:

$$\iint_C \vec{\nabla}\times\vec{F}\bullet d\vec{S} = \iint_C \vec{\nabla}\times\vec{F}\bullet\vec{N}\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4r^3)\,dr\,d\theta = 2\pi\Big[r^4\Big]_0^2 = 32\pi$$

Right Hand Side:

g. Parametrize the circle, ∂C:

$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$

h. Find the tangent vector on the curve:

$\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$

This is counterclockwise. Need clockwise. Reverse $\vec{v} = (2\sin\theta, -2\cos\theta, 0)$

i. Evaluate $\vec{F} = (yz^2, -xz^2, z^3)$ on the circle:

$\vec{F}\Big|_{\vec{r}(\theta)} = (8\sin\theta, -8\cos\theta, 8)$

j. Compute the dot product of $\vec{F}$ and the tangent vector $\vec{v}$:

$\vec{F} \cdot \vec{v} = 16\sin^2\theta + 16\cos^2\theta = 16$

k. Compute the right hand side:

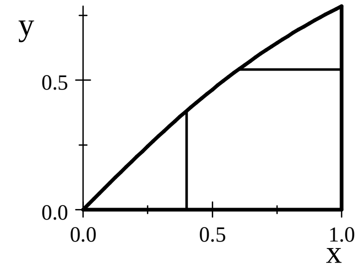
$\int_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 16\, d\theta = 32\pi$ which agrees with (f).

12. (5 points) Sketch the region of integration for the integral

$$\int_0^{\pi/4}\int_{\tan(y)}^{1} \frac{1}{\arctan(x)}\,dx\,dy.$$

Shade in the region.

You computed its value in problem (4).



13. (20 points) A cardboard box needs to hold 18 cm^3. The cardboard for the vertical sides costs 12¢ per cm^2 while the thicker bottom costs 16¢ per cm^2. There is no top. What are the length, width, height and cost of the box which costs the least?

Solution: The volume constraint is $V = LWH = 18$. The cost is $C = 16LW + 12(2LH + 2WH) = 16LW + 24LH + 24WH$ which needs to be minimized.

Lagrange Multipliers:

$\vec{\nabla}C = (16W + 24H, 16L + 24H, 24L + 24W)$ $\quad$ $\vec{\nabla}V = (WH, LH, LW)$ $\quad$ $\vec{\nabla}C = \lambda\vec{\nabla}V$

$16W + 24H = \lambda WH$ $\quad$ $16LW + 24LH = \lambda LWH$

$16L + 24H = \lambda LH$ $\quad$ $16LW + 24WH = \lambda LWH$ $\quad$ $16LW + 24LH = 16LW + 24WH$

$24LH = 24WH$ $\quad$ $L = W$

$24L + 24W = \lambda LW$ $\quad$ $24LH + 24WH = \lambda LWH$ $\quad$ $16LW + 24WH = 24LH + 24WH$

$16LW = 24LH$ $\quad$ $2W = 3H$

$LWH = WW\frac{2}{3}W = 18$ $\quad$ $W^3 = 27$ $\quad$ $W = 3$ $\quad$ $L = 3$ $\quad$ $H = 2$

$C = 16LW + 24LH + 24WH = 16 \cdot 3 \cdot 3 + 24 \cdot 3 \cdot 2 + 24 \cdot 3 \cdot 2 = 432$¢