1-9	/63
10	/20
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Name________________________

MATH 251 Exam 1A Fall 2016

Sections 504 Solutions P. Yasskin

Multiple Choice: (7 points each. No part credit.)

1. Find the distance from the point $\langle 3,4,12\rangle$ to the sphere $x^2+y^2+z^2=64$.

a. 1

b. 5 Correct

c. 8

d. 13

e. 105

Solution: The distance from $\langle 3,4,12\rangle$ to the origin is $\sqrt{3^2+4^2+12^2}=13$. The radius of the sphere is $R=8$. So the point is 5 units outside the sphere.

2. Find a and b so that $a(1,2)+b(2,1)=(0,3)$. What is $a+b$?

a. 1 Correct

b. 2

c. 3

d. 4

e. 5

Solution: $a+2b=0 \qquad 2a+b=3$

The 1^{st} equation says $a=-2b$. So the 2^{nd} equation says $-4b+b=3$ or $b=-1$ and $a=2$. So $a+b=1$.

3. The plot at the right is which polar curve?

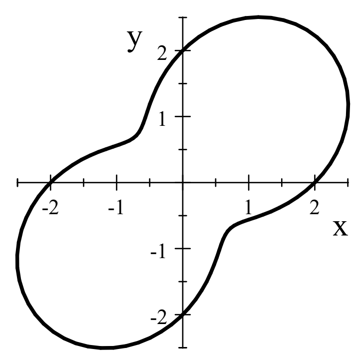
a. $r=2-\cos(2\theta)$

b. $r=2+\cos(2\theta)$

c. $r=2-\sin(2\theta)$

d. $r=2+\sin(2\theta)$ Correct

e. $r=\theta$



Solution: From the plot, when $\theta=0$, we have $r=2$, which is only true for equations (c) and (d). When $\theta=\frac{\pi}{4}$, we have $r=3$, which is only true for equation (d).

4. Find a vector perpendicular to the plane thru the points $P = (2,3,0)$, $Q = (4,-1,-1)$ and $R = (2,0,2)$.

a. $\langle 11,-4,-6\rangle$
b. $\langle -11,3,-2\rangle$
c. $\langle -11,-4,-6\rangle$ Correct
d. $\langle -11,-3,-2\rangle$
e. $\langle -11,4,-6\rangle$

Solution: $\overrightarrow{PQ} = Q - P = \langle 2,-4,-1\rangle$ $\overrightarrow{PR} = R - P = \langle 0,-3,2\rangle$ and

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & -1 \\ 0 & -3 & 2 \end{vmatrix} = \hat{\imath}(-8-3) - \hat{\jmath}(4-0) + \hat{k}(-6) = \langle -11,-4,-6\rangle$$

5. A triangle has vertices at $P = (1,0,4)$, $Q = (1,0,2)$ and $R = \left(2,\sqrt{3},0\right)$. Find the angle at Q.

a. $30°$
b. $45°$
c. $60°$
d. $120°$
e. $135°$ Correct

Solution: $\overrightarrow{QP} = (0,0,2)$ $\overrightarrow{QR} = \left(1,\sqrt{3},-2\right)$ $\left|\overrightarrow{QP}\right| = \sqrt{4} = 2$ $\left|\overrightarrow{QR}\right| = \sqrt{1+3+4} = \sqrt{8} = 2\sqrt{2}$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = -4 \qquad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \frac{-4}{2 \cdot 2\sqrt{2}} = \frac{-1}{\sqrt{2}} \qquad \theta = 135°$$

6. Find the plane tangent to the graph of the function $z = f(x,y) = x^2\sin(y) + x\cos(y)$ at the point $(x,y) = (2,\pi)$. Its z-intercept is

a. 4π Correct
b. 2π
c. 2
d. -4π
e. -2π

Solution: $f(2,\pi) = 4\sin(\pi) + 2\cos(\pi) = -2$
$f_x(x,y) = 2x\sin(y) + \cos(y)$ $\quad f_x(2,\pi) = 4\sin(\pi) + \cos(\pi) = -1$
$f_y(x,y) = x^2\cos(y) - x\sin(y)$ $\quad f_y(2,\pi) = 4\cos(\pi) - 2\sin(\pi) = -4$
$z = f(2,\pi) + f_x(2,\pi)(x-2) + f_y(2,\pi)(y-\pi) = -2 - 1(x-2) - 4(y-\pi) = -x - 4y + 4\pi$

7. A plane is flying from WEST to EAST, directly over the equator at a constant altitude of 100 kilometers above sea level. (Since the Earth is a sphere, the path of the plane is part of a great circle.) In what direction do $\hat{N}$ and $\hat{B}$ point?

a. $\hat{N}$ points SOUTH and $\hat{B}$ points DOWN
b. $\hat{N}$ points SOUTH and $\hat{B}$ points UP
c. $\hat{N}$ points DOWN and $\hat{B}$ points NORTH Correct
d. $\hat{N}$ points DOWN and $\hat{B}$ points SOUTH
e. $\hat{N}$ points UP and $\hat{B}$ points NORTH

Solution: $\hat{T}$ points EAST. Since the path is a circle the aceleration points toward the center. So $\hat{N}$ points toward the center of the Earth which is DOWN. $\hat{B} = \hat{T} \times \hat{N}$ which points NORTH.

8. Find the mass of a wire in the shape of the semi-circle $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ for $0 \le \theta \le \pi$ if the linear density is given by $\delta = y$.

a. π
b. 3π
c. 6
d. 12
e. 18 Correct

Solution: The tangent vector is $\vec{v} = (-3\sin\theta, 3\cos\theta)$ and its length is $|\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta} = 3$. The density along the curve is $\delta(\vec{r}(t)) = 3\sin\theta$. So the mass is:
$M = \int_0^\pi \delta\, ds = \int_0^\pi \delta(\vec{r}(t))\, |\vec{v}|\, d\theta = \int_0^\pi 3\sin\theta\, 3\, d\theta = \left[-9\cos\theta\right]_0^\pi = 9 - -9 = 18.$

9. A bead is pushed along a wire in the shape of the twisted cubic $\vec{r}(t) = (t^2, t^3, t)$ by the force $\vec{F} = \langle x, z, -y \rangle$ from $(1,1,1)$ to $(4,8,2)$. Find the work done.

a. 15 Correct
b. 16
c. $\frac{45}{2}$
d. 45
e. 48

Solution: $\vec{v} = \langle 2t, 3t^2, 1 \rangle \quad \vec{F}(\vec{r}(t)) = \langle t^2, t, -t^3 \rangle \quad \vec{F} \cdot \vec{v} = 2t^3 + 3t^3 - t^3 = 4t^3$
$W = \int \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v}\, dt = \int_1^2 4t^3\, dt = [t^4]_1^2 = 16 - 1 = 15$

Work Out: (20 points each. Part credit possible. Show all work.)

10. Find a parametric equation for the line of intersection of the two planes

$$2x - y + 3z = 7 \quad \text{and} \quad 3x + y + 2z = 3$$

HINTS: Find the normal vectors, $\vec{N}_1$ and $\vec{N}_2$, to the 2 planes, the direction vector, $\vec{v}$, of the line of intersection and any one point, P, on the intersection.

Solution: The normals are $\vec{N}_1 = \langle 2,-1,3\rangle$ and $\vec{N}_2 = \langle 3,1,2\rangle$. The direction vector is

$$\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = \hat{\imath}(-2-3) - \hat{\jmath}(4-9) + \hat{k}(2+3) = \langle -5,5,5\rangle$$

We look for a point with $z = 0$. So we solve $2x - y = 7$ and $3x + y = 3$.
The sum is $5x = 10$. So $x = 2$. Then the first equation gives $4 - y = 7$. So $y = -3$.
So a point of intersection is $P = (2,-3,0)$. Then the line is
$X = P + t\vec{v}$ or $(x,y,z) = (2,-3,0) + t\langle -5,5,5\rangle$ or $(x,y,z) = (2-5t,-3+5t,5t)$
or $x = 2-5t$, $y = -3+5t$, $z = 5t$

11. As Duke Skywater flies the Century Eagle through the galaxy he wants to maximize the Power of the Force which is given by $F = \dfrac{1}{D}$ where D is the dark matter density given by $D = x^2 + y^2 + z^2$.
If his current position is $\vec{r} = (1,2,1)$ and his current velocity is $\vec{v} = (0.2, 0.5, -0.3)$, what is the current rate of change of the Power of the Force, $\dfrac{dF}{dt}$?
(Plug in numbers but you don't need to simplify.)

Solution: The position says $x = 1$, $y = 2$, $z = 1$.
The velocity says $\dfrac{dx}{dt} = 0.2$, $\dfrac{dy}{dt} = 0.5$, $\dfrac{dz}{dt} = -0.3$.
Currently $D = x^2 + y^2 + z^2 = 1^2 + 2^2 + 1^2 = 6$.
We use the chain rule twice:

$$\frac{dF}{dt} = \frac{dF}{dD}\frac{dD}{dt} = \frac{dF}{dD}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) = \frac{-1}{D^2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$

$$= \frac{-1}{36}(2\cdot 1\cdot(0.2) + 2\cdot 2\cdot(0.5) + 2\cdot 1\cdot(-0.3)) = \frac{-1.8}{36} = -0.05$$