Name________________________

MATH 251 Exam 1B Fall 2016

Sections 504 Solutions P. Yasskin

1-9	/63
10	/20
11	/20
Total	/103

Multiple Choice: (7 points each. No part credit.)

1. Find the distance from the point $\langle 3,4,12\rangle$ to the sphere $x^2+y^2+z^2=36$.

a. 1
b. 6
c. 7 Correct
d. 13
e. 105

Solution: The distance from $\langle 3,4,12\rangle$ to the origin is $\sqrt{3^2+4^2+12^2}=13$. The radius of the sphere is $R=6$. So the point is 7 units outside the sphere.

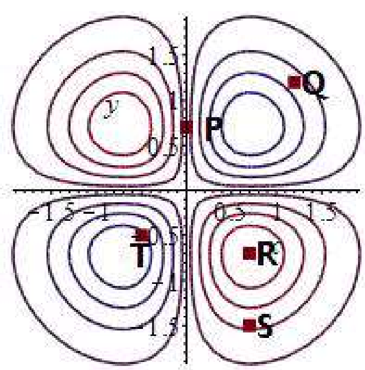
2. Find a and b so that $a(1,2)-b(2,1)=(0,3)$. What is $a+b$?

a. 1
b. 2
c. 3 Correct
d. 4
e. 5

Solution: $a-2b=0 \qquad 2a-b=3$

The 1^{st} equation says $a=2b$. So the 2^{nd} equation says $4b-b=3$ or $b=1$ and $a=2$. So $a+b=3$.

3. In the plot at the right, which point could be a local maximum?



a. $P=\left(0,\dfrac{1}{\sqrt{2}}\right)$
b. $Q=\left(\sqrt{2},\sqrt{2}\right)$
c. $R=\left(\dfrac{1}{\sqrt{2}},\dfrac{-1}{\sqrt{2}}\right)$ Correct
d. $S=\left(\dfrac{1}{\sqrt{2}},-\sqrt{2}\right)$
e. $T=\left(\dfrac{-1}{2},\dfrac{-1}{2}\right)$

Solution: Near a local maximum, the contours form circles around the local maximum. So R is the local maximum.

4. Find a vector perpendicular to the plane thru the points $P = (2,3,-1)$, $Q = (4,-1,0)$ and $R = (2,0,2)$.

a. $\langle 9,-6,-6\rangle$
b. $\langle -9,5,5\rangle$
c. $\langle -9,-5,5\rangle$
d. $\langle -9,-6,-6\rangle$ Correct
e. $\langle -9,6,-6\rangle$

Solution: $\overrightarrow{PQ} = Q - P = \langle 2,-4,1\rangle$ $\overrightarrow{PR} = R - P = \langle 0,-3,3\rangle$ and

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -4 & 1 \\ 0 & -3 & 3 \end{vmatrix} = \hat{\imath}(-12+3) - \hat{\jmath}(6-0) + \hat{k}(-6-0) = \langle -9,-6,-6\rangle$$

5. A triangle has vertices at $P = (-2,1,0)$, $Q = (-1,1,1)$ and $R = (1,3,1)$. Find the angle at Q.

a. $30°$
b. $45°$
c. $60°$
d. $120°$ Correct
e. $135°$

Solution: $\overrightarrow{QP} = (-1,0,-1)$ $\overrightarrow{QR} = (2,2,0)$ $\left|\overrightarrow{QP}\right| = \sqrt{2}$ $\left|\overrightarrow{QR}\right| = \sqrt{8} = 2\sqrt{2}$

$$\overrightarrow{QP} \cdot \overrightarrow{QR} = -2 \qquad \cos\theta = \frac{\overrightarrow{QP} \cdot \overrightarrow{QR}}{\left|\overrightarrow{QP}\right|\left|\overrightarrow{QR}\right|} = \frac{-2}{\sqrt{2}\,2\sqrt{2}} = \frac{-1}{2} \qquad \theta = 120°$$

6. Find the plane tangent to the graph of the function $z = f(x,y) = x^2\sin(y) - x\cos(y)$ at the point $(x,y) = (2,\pi)$. Its z-intercept is

a. -4π
b. -2π
c. 2
d. 2π
e. 4π Correct

Solution: $f(2,\pi) = 4\sin(\pi) - 2\cos(\pi) = 2$
$f_x(x,y) = 2x\sin(y) - \cos(y)$ $f_x(2,\pi) = 4\sin(\pi) - \cos(\pi) = 1$
$f_y(x,y) = x^2\cos(y) + x\sin(y)$ $f_y(2,\pi) = 4\cos(\pi) + 2\sin(\pi) = -4$
$z = f(2,\pi) + f_x(2,\pi)(x-2) + f_y(2,\pi)(y-\pi) = 2 + 1(x-2) - 4(y-\pi) = x - 4y + 4\pi$

7. A plane is flying from EAST to WEST, directly over the equator at a constant altitude of 100 kilometers above sea level. (Since the Earth is a sphere, the path of the plane is part of a great circle.) In what direction do $\hat{N}$ and $\hat{B}$ point?

a. $\hat{N}$ points SOUTH and $\hat{B}$ points DOWN
b. $\hat{N}$ points SOUTH and $\hat{B}$ points UP
c. $\hat{N}$ points DOWN and $\hat{B}$ points NORTH
d. $\hat{N}$ points DOWN and $\hat{B}$ points SOUTH Correct
e. $\hat{N}$ points UP and $\hat{B}$ points NORTH

Solution: $\hat{T}$ points WEST. Since the path is a circle the aceleration points toward the center. So $\hat{N}$ points toward the center of the Earth which is DOWN. $\hat{B} = \hat{T} \times \hat{N}$ which points SOUTH.

8. Find the mass of a wire in the shape of the semi-circle $\vec{r}(\theta) = (3\cos\theta, 3\sin\theta)$ for $0 \le \theta \le \pi$ if the linear density is given by $\delta = y$.

a. 18 Correct
b. 12
c. 6
d. 3π
e. π

Solution: The tangent vector is $\vec{v} = (-3\sin\theta, 3\cos\theta)$ and its length is $|\vec{v}| = \sqrt{9\sin^2\theta + 9\cos^2\theta} = 3$. The density along the curve is $\delta(\vec{r}(t)) = 3\sin\theta$. So the mass is:
$M = \int_0^\pi \delta\, ds = \int_0^\pi \delta(\vec{r}(t))\, |\vec{v}|\, d\theta = \int_0^\pi 3\sin\theta\, 3\, d\theta = \left[-9\cos\theta\right]_0^\pi = 9 - -9 = 18.$

9. A bead is pushed along a wire in the shape of the twisted cubic $\vec{r}(t) = (t^3, t^2, t)$ by the force $\vec{F} = \langle z^3, yz^2, xz^2 \rangle$ from $(1,1,1)$ to $(8,4,2)$. Find the work done.

a. 186
b. $\frac{384}{7}$
c. $\frac{381}{7}$
d. 63 Correct
e. 64

Solution: $\vec{v} = \langle 3t^2, 2t, 1 \rangle$ $\quad \vec{F}(\vec{r}(t)) = \langle t^3, t^4, t^5 \rangle$ $\quad \vec{F} \cdot \vec{v} = 3t^5 + 2t^5 + t^5 = 6t^5$
$W = \int \vec{F} \cdot d\vec{s} = \int_1^2 \vec{F} \cdot \vec{v}\, dt = \int_1^2 6t^5\, dt = [t^6]_1^2 = 64 - 1 = 63$

Work Out: (20 points each. Part credit possible. Show all work.)

10. Find a parametric equation for the line of intersection of the two planes

$$2x + y + 3z = 7 \quad \text{and} \quad 3x - y + 2z = 3$$

HINTS: Find the normal vectors, $\vec{N}_1$ and $\vec{N}_2$, to the 2 planes, the direction vector, $\vec{v}$, of the line of intersection and any one point, P, on the intersection.

Solution: The normals are $\vec{N}_1 = \langle 2,1,3 \rangle$ and $\vec{N}_2 = \langle 3,-1,2 \rangle$. The direction vector is

$$\vec{v} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & 3 \\ 3 & -1 & 2 \end{vmatrix} = \hat{\imath}(2+3) - \hat{\jmath}(4-9) + \hat{k}(-2-3) = \langle 5,5,-5 \rangle$$

We look for a point with $z = 0$. So we solve $2x + y = 7$ and $3x - y = 3$.
The sum is $5x = 10$. So $x = 2$. Then the first equation gives $4 + y = 7$. So $y = 3$.
So a point of intersection is $P = (2,3,0)$. Then the line is
$X = P + t\vec{v}$ or $(x,y,z) = (2,3,0) + t\langle 5,5,-5 \rangle$ or $(x,y,z) = (2+5t, 3+5t, -5t)$
or $x = 2 + 5t$, $y = 3 + 5t$, $z = -5t$

11. As Duke Skywater flies the Century Eagle through the galaxy he wants to maximize the Power of the Force which is given by $F = \frac{1}{D}$ where D is the dark matter density given by $D = x^2 + y^2 + z^2$.
If his current position is $\vec{r} = (2,1,1)$ and his current velocity is $\vec{v} = (0.6,-0.2,0.8)$, what is the current rate of change of the Power of the Force, $\frac{dF}{dt}$?
(Plug in numbers but you don't need to simplify.)

Solution: The position says $x = 2$, $y = 1$, $z = 1$.
The velocity says $\frac{dx}{dt} = 0.6$, $\frac{dy}{dt} = -0.2$, $\frac{dz}{dt} = 0.8$.
Currently $D = x^2 + y^2 + z^2 = 2^2 + 1^2 + 1^2 = 6$.
We use the chain rule twice:

$$\frac{dF}{dt} = \frac{dF}{dD}\frac{dD}{dt} = \frac{dF}{dD}\left(\frac{\partial D}{\partial x}\frac{dx}{dt} + \frac{\partial D}{\partial y}\frac{dy}{dt} + \frac{\partial D}{\partial z}\frac{dz}{dt}\right) = \frac{-1}{D^2}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$

$$= \frac{-1}{36}(2 \cdot 2 \cdot (0.6) + 2 \cdot 1 \cdot (-0.2) + 2 \cdot 1 \cdot (0.8)) = \frac{-3.6}{36} = -0.1$$