1-9	/63
10	/20
11	/24
Total	/107

Name__________________________

MATH 251 Exam 2A Fall 2016

Sections 504 Solutions P. Yasskin

Multiple Choice: (7 points each. No part credit.)

1. The function $f = xy - \frac{2}{x} + \frac{4}{y}$ has a critical point at $(x,y) = (1,-2)$.
Use the Second Derivative Test to classify this critical point.

a. Local Minimum
b. Local Maximum Correct
c. Inflection Point
d. Saddle Point
e. Test Fails

Solution: $f_x = y + \frac{2}{x^2}$ $f_y = x - \frac{4}{y^2}$

$f_{xx} = -\frac{4}{x^3}$ $f_{yy} = \frac{8}{y^3}$ $f_{xy} = 1$

$f_{xx}(1,-2) = -4$ $f_{yy}(1,-2) = -1$ $f_{xy}(1,-2) = 1$ $D = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3$

Local Maximum

2. Find the volume of the solid under $z = 2x^2y$ above the region in the xy-plane between $y = x$ and $y = x^2$.

a. $\frac{2}{35}$ Correct
b. $\frac{35}{12}$
c. $\frac{12}{35}$
d. $\frac{1}{35}$
e. $\frac{1}{12}$

Solution: $V = \iint 2x^2y\,dA = \int_0^1 \int_{x^2}^{x} 2x^2y\,dy\,dx = \int_0^1 \left[x^2y^2\right]_{y=x^2}^{x} dx$

$$= \int_0^1 (x^4 - x^6)\,dx = \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_{x=0}^{1} = \frac{1}{5} - \frac{1}{7} = \frac{7-5}{35} = \frac{2}{35}$$

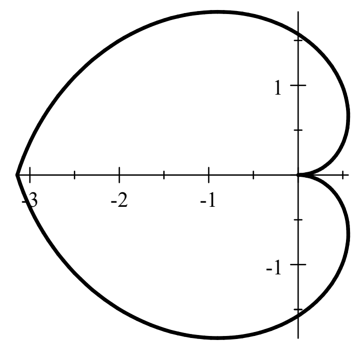
3. Compute $\iint \sin(x^2)\,dx\,dy$ over the triangle with vertices $(0,0)$, $(\sqrt{\pi},0)$, $(\sqrt{\pi},\sqrt{\pi})$.

a. $-\pi$
b. $-\sqrt{\pi}$
c. 1 Correct
d. $\sqrt{\pi}$
e. π

Solution: You must do the y-integral first because you don't know the antiderivative of $\sin(x^2)$. The edges are $y = 0$, $x = \sqrt{\pi}$, $y = x$.

$$\iint \sin(x^2)\,dx\,dy = \int_0^{\sqrt{\pi}} \int_0^x \sin(x^2)\,dy\,dx = \int_0^{\sqrt{\pi}} \Big[y \sin(x^2) \Big]_{y=0}^{x} dx = \int_0^{\sqrt{\pi}} x \sin(x^2)\,dx$$
$$= \left[\frac{-1}{2} \cos(x^2) \right]_{x=0}^{\sqrt{\pi}} = \frac{1}{2} - -\frac{1}{2} = 1$$

4. Find the area of the heart shaped region inside the polar curve $r = |\theta|$.



a. $\dfrac{\pi^3}{6}$
b. $\dfrac{\pi^3}{3}$ Correct
c. $\dfrac{4\pi^3}{3}$
d. $\dfrac{8\pi^3}{3}$
e. $\dfrac{16\pi^3}{3}$

Solution: Double the upper half:

$$A = 2\iint 1\,dA = 2\int_0^{\pi} \int_0^{\theta} r\,dr\,d\theta = 2\int_0^{\pi} \left[\frac{r^2}{2} \right]_{r=0}^{\theta} d\theta = 2\int_0^{\pi} \left(\frac{\theta^2}{2} \right) d\theta = 2\left[\frac{\theta^3}{6} \right]_{\theta=0}^{\pi} = \frac{\pi^3}{3}$$

5. The solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ has density $\delta = z$. Find the total mass.

a. $\pi/2$
b. π
c. 2π
d. 4π Correct
e. 8π

Solution: In spherical coordinates, $\delta = z = \rho\cos\varphi$ and $J = \rho^2 \sin\varphi$.

$$M = \iiint \delta\,dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^2 \rho\cos\varphi\,\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta = 2\pi \left[\frac{\rho^4}{4} \right]_{\rho=0}^{2} \left[\frac{\sin^2\varphi}{2} \right]_{\varphi=0}^{\pi/2} = 4\pi$$

6. The solid hemisphere $0 \le z \le \sqrt{4 - x^2 - y^2}$ has density $\delta = z$.

Find the z-component of the center of mass.

a. 1

b. $\dfrac{32}{15}\pi$

c. $\dfrac{64}{15}\pi$

d. $\dfrac{8}{15}$

e. $\dfrac{16}{15}$ Correct

Solution: $M_z = \iiint z\delta\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} \rho^2\cos^2\varphi\,\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta = 2\pi\left[\dfrac{\rho^5}{5}\right]_{\rho=0}^{2}\left[\dfrac{-\cos^3\varphi}{3}\right]_{\varphi=0}^{\pi/2} = \dfrac{64}{15}\pi$

$\bar{z} = \dfrac{M_z}{M} = \dfrac{64\pi}{15 \cdot 4\pi} = \dfrac{16}{15}$

7. Compute $\iiint \nabla \cdot \vec{F}\, dV$ on the solid cylinder bounded by

$x^2 + y^2 = 9,\quad z = 0$ and $z = 5$ for the vector field $\vec{F} = (x^3, y^3, z(x^2+y^2))$.

a. 45π

b. 90π

c. 360π

d. 810π Correct

e. 900π

Solution: $\nabla \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4x^2 + 4y^2 = 4r^2$

$\iiint \nabla \cdot \vec{F}\, dV = \int_0^5\int_0^{2\pi}\int_0^3 4r^2\, r\, dr\, d\theta\, dz = 5 \cdot 2\pi\left[r^4\right]_{r=0}^{3} = 810\pi$

8. If $\vec{F} = (-yz, xz, z^2)$, compute $\vec{F} \cdot \vec{\nabla} \times \vec{F}$.

a. z^3

b. $z^3 - xyz$

c. $2z^3$ Correct

d. $2z^3 - 2xyz$

e. 0

Solution:

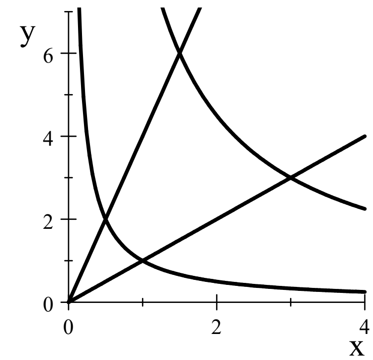
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -yz & xz & z^2 \end{vmatrix} = \hat{\imath}(x) - \hat{\jmath}(-y) + \hat{k}(z - -z) = (x, y, 2z)$$

$\vec{F} \cdot \vec{\nabla} \times \vec{F} = -yzx + xzy + z^2 2z = 2z^3$

9. Compute $\iint_R x^2\, dx\,dy$ over the diamond shaped region R bounded by

$$y = \frac{1}{x}, \qquad y = \frac{9}{x}, \qquad y = x, \qquad y = 4x$$

HINT: Use the curvilinear coordinates (u,v) where $x = uv$ and $y = \frac{u}{v}$.



a. -15

b. $-\frac{91}{36}$

c. $\frac{91}{6}$

d. $\frac{91}{36}$

e. 15 Correct

Solution: $J = \left|\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\right| = \left|\left|\begin{matrix} v & \frac{1}{v} \\ u & -\frac{u}{v^2}\end{matrix}\right|\right| = \left|-\frac{u}{v} - \frac{u}{v}\right| = \frac{2u}{v}$

Boundaries: $xy = 1 \Rightarrow u^2 = 1 \Rightarrow u = 1 \qquad xy = 9 \Rightarrow u^2 = 9 \Rightarrow u = 3$

$\frac{y}{x} = 1 \Rightarrow \frac{1}{v^2} = 1 \Rightarrow v = 1 \qquad \frac{y}{x} = 4 \Rightarrow \frac{1}{v^2} = 4 \Rightarrow v = \frac{1}{2}$

$$\iint_R x^2\, dx\,dy = \int_{1/2}^{1}\int_1^3 u^2v^2\,\frac{2u}{v}\,du\,dv = 2\int_{1/2}^{1}\int_1^3 u^3v\,du\,dv$$

$$= 2\left[\frac{u^4}{4}\right]_{u=1}^{3}\left[\frac{v^2}{2}\right]_{v=1/2}^{1} = 2\left[\frac{81}{4} - \frac{1}{4}\right]\left[\frac{1}{2} - \frac{1}{8}\right] = 2(20)\left(\frac{3}{8}\right) = 15$$

Work Out: (Points indicated. Part credit possible. Show all work.)

10. (20 points) Find the point $P = (x,y,z)$ on the plane $z = x + y - 3$ which is closest to the point $Q = (1,0,1)$. Find the distance from P to Q.

Solution: Minimize the distance $D = \sqrt{(x-1)^2 + y^2 + (z-1)^2}$ or its square:

$f = D^2 = (x-1)^2 + y^2 + (z-1)^2$ subject to the constraint $z = x + y - 3$.

So $f = (x-1)^2 + y^2 + (x+y-4)^2$

$f_x = 2(x-1) + 2(x+y-4) = 0 \Rightarrow 4x + 2y - 10 = 0 \Rightarrow 2x + y = 5$ (a)

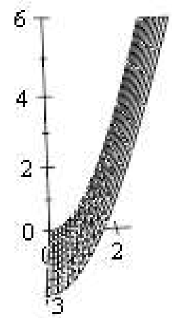
$f_y = 2y + 2(x+y-4) = 0 \Rightarrow 2x + 4y - 8 = 0 \Rightarrow x + 2y = 4$ (b)

(a)-2*(b): $-3y = -3 \Rightarrow y = 1 \Rightarrow x = 2 \Rightarrow z = 0$

$P = (2,1,0)$

$D = \sqrt{(2-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$

11. (24 points) Consider the sliding board given by $z = y^2$ for $0 \le x \le 3,\ \ y \ge 0$ and $z \le 6$.



- Find the mass of the slide if the surface mass density is $\delta = y$.
- If the slide is porous and rain is falling on the slide with velocity field $\vec{V} = (0, x, -y)$ find the flux of $\vec{V}$ **down** through the slide.

Parametrize the slide as $\vec{R}(u,v) = (u, v, v^2)$ and follow these steps:

a. Find the coordinate tangent vectors:

$$\begin{vmatrix} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_u = & (1 & 0 & 0) \\ \vec{e}_v = & (0 & 1 & 2v) \end{vmatrix}$$

b. Find the normal vector and check its orientation:

$\vec{N} = i(0) - j(2v) + k(1) = (0, -2v, 1)$

This is up. We need down. So we reverse the normal.

After checking orientation:

$\vec{N} = (0, 2v, -1)$

c. Find the length of the normal vector:

$\left|\vec{N}\right| = \sqrt{4v^2 + 1}$

d. Find the limits on u and v:

$u = x$ So $0 \le u \le 3$. $v = y = \sqrt{z}$ So $0 \le v \le \sqrt{6}$.

(continued)

e. Evaluate the density $\delta = y$ on the slide:

$$\delta\left(\vec{R}(u,v)\right) = v$$

f. Compute the mass:

$$M = \iint \delta\, dS = \int_0^3 \int_0^{\sqrt{6}} v\sqrt{4v^2 + 1}\; dv\, du = \Big[\, u \,\Big]_{u=0}^{3} \left[\frac{2}{3 \cdot 8} (4v^2 + 1)^{3/2} \right]_{v=0}^{\sqrt{6}}$$

$$= \frac{1}{4}(25^{3/2} - 1) = \frac{124}{4} = 31$$

g. Evaluate the velocity field $\vec{V} = (0, x, -y)$ on the slide:

$$\vec{V}\left(\vec{R}(u,v)\right) = (0, u, -v)$$

h. Compute the flux:

$$\iint \vec{V} \cdot d\vec{S} = \int_0^3 \int_0^{\sqrt{6}} \vec{V} \cdot \vec{N}\, dv\, du = \int_0^3 \int_0^{\sqrt{6}} (2uv + v)\, dv\, du = \int_0^3 (2u + 1)\, du \int_0^{\sqrt{6}} (v)\, dv$$

$$= \Big[\, u^2 + u \,\Big]_{u=0}^{3} \left[\frac{v^2}{2} \right]_{v=0}^{\sqrt{6}} = (12)(3) = 36$$