1-13	/65	16	/10
14	/10	E.C.	/ 5
15	/20	Total	/110

Name______________________________

MATH 251 Final Exam Version A Fall 2016

Sections 504 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the point where the line $x = 2 + t$, $y = 1 - t$, $z = -2 + 2t$ intersects the plane $2x - y - z = 6$. At this point $x + y + z =$

- **a**. 0
- **b**. 1
- **c**. 2
- **d**. 3 Correct Choice
- **e**. 4

$2x - y - z = 2(2 + t) - (1 - t) - (-2 + 2t) = t + 5 = 6 \qquad t = 1$

$x = 3,\ \ y = 0,\ \ z = 0, \qquad x + y + z = 3$

2. Find the plane tangent to the graph of $z = x^2 y$ at the point $(2,3)$. The z-intercept is

- **a**. -24 Correct Choice
- **b**. -16
- **c**. -4
- **d**. 0
- **e**. 4

$f(x,y) = x^2 y \qquad f_x(x,y) = 2xy \qquad f_y(x,y) = x^2$

$f(2,3) = 12 \qquad f_x(2,3) = 12 \qquad f_y(2,3) = 4$

Tan plane: $z = 12 + 12(x - 2) + 4(y - 3) = 12x + 4y - 24 \qquad z\text{-intercept} = -24$

3. Find the line perpendicular to the hyperboloid $z^2 - x^2 - y^2 = 3$ at the point $P = (2,3,4)$. This line intersects the xy-plane at

- **a**. $(6,-8,0)$
- **b**. $(-2,-3,0)$
- **c**. $(8,6,0)$
- **d**. $(4,6,0)$ Correct Choice
- **e**. $\left(\frac{8}{3},2,0\right)$

$F = z^2 - x^2 - y^2 \qquad \vec{\nabla} F = (-2x,-2y,2z) \qquad \vec{N} = \vec{\nabla} F\Big|_{(2,3,4)} = (-4,-6,8)$

$X = P + t\vec{N} = (2,3,4) + t(-4,-6,8) = (2 - 4t, 3 - 6t, 4 + 8t)$

Intersects the xy-plane when $z = 0$ or $4 + 8t = 0$ or $t = -\frac{1}{2}$.

So $x = 2 - 4\left(-\frac{1}{2}\right) = 4$ and $y = 3 - 6\left(-\frac{1}{2}\right) = 6$

4. A box currently has length $L = 20$ cm which is increasing at 4 cm/sec, width $W = 15$ cm which is decreasing at 2 cm/sec, and height $H = 12$ cm which is increasing at 1 cm/sec. At what rate is the volume changing?

a. 3 cm^3/sec
b. 7 cm^3/sec
c. 540 cm^3/sec   Correct Choice
d. 900 cm^3/sec
e. 3600 cm^3/sec

$$V = LWH \qquad \frac{dV}{dt} = WH\frac{dL}{dt} + LH\frac{dW}{dt} + LW\frac{dH}{dt} = 15 \cdot 12 \cdot 4 - 20 \cdot 12 \cdot 2 + 20 \cdot 15 \cdot 1 = 540$$

5. In order to hide from the Dark Invader, Duke Skywater flys the Millenium Eagle into a galactic dust storm. Currently, his position is $P = (10,-20,30)$ and his velocity is $\vec{v} = (-4,9,2)$. He measures that currently the dust density is $\rho = 520$ and its gradient is $\vec{\nabla}\rho = (2,1,-2)$. Find the current rate of change of the dust density as seen by Duke.

a. -60
b. 60
c. 517
d. 523
e. -3 Correct Choice

$$\vec{\nabla}_{\vec{v}}\rho = \vec{v} \cdot \vec{\nabla}\rho = (-4,9,2) \cdot (2,1,-2) = -8 + 9 - 4 = -3$$

6. Under the same conditions as in #5, in what **unit** vector direction should Duke travel to **increase** the dust density as quickly as possible?

a. $(2,1,-2)$
b. $(-2,-1,2)$
c. $\left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$ Correct Choice
d. $\left(\frac{2}{3},\frac{-1}{3},\frac{-2}{3}\right)$
e. $\left(\frac{-2}{3},\frac{-1}{3},\frac{2}{3}\right)$

$$\hat{u} = \frac{\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \frac{(2,1,-2)}{\sqrt{4+1+4}} = \left(\frac{2}{3},\frac{1}{3},\frac{-2}{3}\right)$$

7. The point $(2,-1)$ is a critical point of the function $f = \dfrac{x^2y^2 + 2x - 4y}{xy}$.
Use the Second Derivative Test to classify the point.

- **a**. Local Minimum
- **b**. Local Maximum   Correct Choice
- **c**. Inflection Point
- **d**. Saddle Point
- **e**. Test Fails

$$f = xy + \frac{2}{y} - \frac{4}{x} \qquad f_x = y + \frac{4}{x^2} \qquad f_y = x - \frac{2}{y^2}$$

$$f_{xx} = -\frac{8}{x^3} = -1 < 0 \qquad f_{yy} = \frac{4}{y^3} = -4 < 0 \qquad f_{xy} = 1 \qquad D = 4 - 1 = 3 > 0$$ Maximum

8. Find the mass of the plate between $y = 2x$ and $y = x^2$ if the surface density is $\delta = xy$.

- **a**. $\frac{8}{15}$
- **b**. $\frac{16}{15}$
- **c**. $\frac{32}{15}$
- **d**. $\frac{8}{3}$ Correct Choice
- **e**. $\frac{16}{3}$

$$M = \iint \delta\, dA = \int_0^2 \int_{x^2}^{2x} xy\, dy\, dx = \int_0^2 x\left[\frac{y^2}{2}\right]_{x^2}^{2x} dx = \frac{1}{2}\int_0^2 x(4x^2 - x^4)\, dx = \frac{1}{2}\left[x^4 - \frac{x^6}{6}\right]_0^2$$

$$= \frac{1}{2}\left(16 - \frac{32}{3}\right) = \frac{8}{3}$$

9. Find the y-component of the center of mass of the plate between $y = 2x$ and $y = x^2$ if the surface density is $\delta = xy$.

- **a**. $\frac{8}{15}$
- **b**. $\frac{5}{12}$
- **c**. $\frac{12}{5}$ Correct Choice
- **d**. $\frac{15}{8}$
- **e**. $\frac{32}{5}$

$$M_y = \iint y\delta\, dA = \int_0^2 \int_{x^2}^{2x} xy^2\, dy\, dx = \int_0^2 x\left[\frac{y^3}{3}\right]_{x^2}^{2x} dx = \frac{1}{3}\int_0^2 x(8x^3 - x^6)\, dx$$

$$= \frac{1}{3}\left[8\frac{x^5}{5} - \frac{x^8}{8}\right]_0^2 = \frac{1}{3}\left(\frac{2^8}{5} - \frac{2^8}{8}\right) = \frac{2^8}{3}\left(\frac{8-5}{40}\right) = \frac{2^5}{5} = \frac{32}{5}$$

$$\bar{y} = \frac{M_y}{M} = \frac{32}{5}\,\frac{3}{8} = \frac{12}{5}$$

10. Compute $\oint \vec{F} \cdot d\vec{s}$ counterclockwise around the rectangle $0 \le x \le \pi$ and $0 \le y \le 2\pi$ for $\vec{F} = (y\sin x + \sin y, x\cos y + \cos x)$.

HINT: Use the Fundamental Theorem of Calculus for Curves or Green's Theorem.

a. -16π

b. -8π Correct Choice

c. 0

d. 8π

e. 16π

By Green's Theorem: $\vec{F} = (P,Q)$ $P = y\sin x + \sin y$ $Q = x\cos y + \cos x$

$$\oint \vec{F} \cdot d\vec{s} = \iint \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint (\cos y - \sin x - \sin x - \cos y)\,dx\,dy = \int_0^{2\pi} \int_0^{\pi} -2\sin x\,dx\,dy$$
$$= 4\pi \Big[\cos x \Big]_0^{\pi} = -8\pi$$

11. Compute $\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x + y, x + 2y, 2z)$ along the curve $\vec{r}(t) = (\sqrt{t}, t^2 + t^3, t^2 - t^3)$.

HINT: Find a scalar potential.

a. 1

b. 3

c. 5

d. 7 Correct Choice

e. 9

$\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f) = \vec{F} = (2x + y, x + 2y, 2z)$ $\Rightarrow$ $f = x^2 + xy + y^2 + z^2$

By the FTCC, $\int_{(0,0,0)}^{(1,2,0)} \vec{F} \cdot d\vec{s} = \int_{(0,0,0)}^{(1,2,0)} \vec{\nabla} f \cdot d\vec{s} = f(1,2,0) - f(0,0,0) = 7$

12. Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the sphere $x^2 + y^2 + z^2 = 4$ with outward normal for $\vec{F} = (x^2yz, xy^2z, xyz^2)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

a. 4π

b. 12π

c. $\frac{32}{3}\pi$

d. $\frac{64}{3}\pi$

e. 0 Correct Choice

By Stokes' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{s} = 0$ because there is no boundary curve.

By Gauss' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{\nabla} \times \vec{F}\,dV = 0$ because $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$.

13. Compute $\iint \vec{F} \cdot d\vec{S}$ outward through the complete surface of the cylinder $x^2 + y^2 = 4$ for $1 \le z \le 4$ for $\vec{F} = (x^3, y^3, z(x^2 + y^2))$.

HINT: Use a Theorem.

a. 96π Correct Choice

b. 72π

c. 48π

d. 24π

e. 18π

$$\vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + x^2 + y^2 = 4(x^2 + y^2) = 4r^2$$

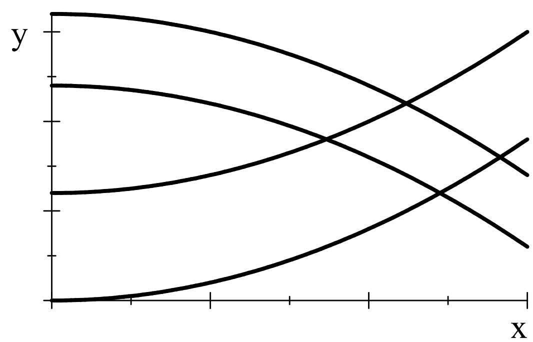
$$\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_1^4 \int_0^{2\pi} \int_0^2 4r^2\, r\, dr\, d\theta\, dz = \Big[z \Big]_1^4 (2\pi)[r^4]_0^2 = (3)(2\pi)16 = 96\pi$$

Work Out: (Points indicated. Part credit possible. Show all work.)

14. (10 points) Compute $\iint_D x\, dx\, dy$ over the "diamond shaped" region D in the first quadrant bounded by the parabolas

$y = 16 - x^2$ $\quad$ $y = 12 - x^2$

$y = 6 + x^2$ and $y = x^2$



HINTS: Use the coordinates: $u = y + x^2$, $v = y - x^2$. Solve for x and y.

$$y = v + x^2 \qquad u = v + 2x^2 \qquad x^2 = \frac{u - v}{2} \qquad \boxed{x = \frac{\sqrt{u - v}}{\sqrt{2}}} \qquad y = v + \frac{u - v}{2} \qquad \boxed{y = \frac{u + v}{2}}$$

$$J = \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \left\| \begin{matrix} \frac{1}{\sqrt{2}} \frac{1}{2\sqrt{u - v}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} \frac{-1}{2\sqrt{u - v}} & \frac{1}{2} \end{matrix} \right\| = \left| \frac{1}{\sqrt{2}} \frac{1}{4\sqrt{u - v}} - - \frac{1}{\sqrt{2}} \frac{1}{4\sqrt{u - v}} \right| = \frac{1}{2\sqrt{2}\sqrt{u - v}}$$

Integrand: $x = \dfrac{\sqrt{u - v}}{\sqrt{2}}$ $\qquad$ Boundaries are:

$y = 16 - x^2 \Rightarrow y + x^2 = 16 \Rightarrow u = 16$ $\qquad$ $y = 12 - x^2 \Rightarrow y + x^2 = 12 \Rightarrow u = 12$

$y = 6 + x^2 \Rightarrow y - x^2 = 6 \Rightarrow v = 6$ $\qquad$ $y = x^2 \Rightarrow y - x^2 = 0 \Rightarrow v = 0$

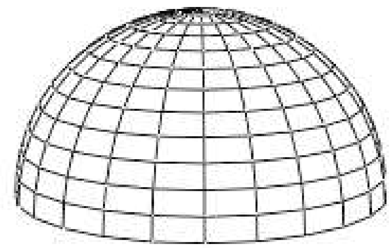
$$\iint_R x\, dx\, dy = \int_0^6 \int_{12}^{16} \frac{\sqrt{u - v}}{\sqrt{2}} \frac{1}{2\sqrt{2}\sqrt{u - v}}\, du\, dv = \int_0^6 \int_{12}^{16} \frac{1}{4}\, du\, dv = \frac{1}{4}[u]_{12}^{16}[v]_0^6 = \frac{1}{4}(4)(6) = 6$$

15. (20 points) Verify Stokes' Theorem $\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial H} \vec{F} \cdot d\vec{s}$

for the vector field $\vec{F} = (-y, x, xz + yz)$

and the hemisphere $z = \sqrt{16 - x^2 - y^2}$ oriented up.

Be sure to check the orientation. Use the following steps:



a. RHS: Parametrize the boundary curve and compute the line integral:

$\vec{r}(\theta) = (4\cos\theta, 4\sin\theta, 0)$

$\vec{v}(\theta) = (-4\sin\theta, 4\cos\theta, 0)$ oriented counterclockwise - OK

$\vec{F}(\vec{r}(\theta)) = (-4\sin\theta, 4\cos\theta, 0)$

$$\oint_{\partial H} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} 16\sin^2\theta + 16\cos^2\theta\, d\theta = 16\int_0^{2\pi} d\theta = 32\pi$$

b. LHS: Compute the surface integral using the parametrization:

$\vec{R}(\varphi, \theta) = (\quad 4\sin\varphi\cos\theta, \quad 4\sin\varphi\sin\theta, \quad 4\cos\varphi \quad)$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & xz + yz \end{vmatrix} = \hat{\imath}(z) - \hat{\jmath}(z) + \hat{k}(1 - -1) = (z, -z, 2)$$

$\left[\vec{\nabla} \times \vec{F}\right]_{\vec{R}(\varphi,\theta)} = (4\cos\varphi, -4\cos\varphi, 2)$

$\vec{e}_\varphi = (\quad 4\cos\varphi\cos\theta, \quad 4\cos\varphi\sin\theta, \quad -4\sin\varphi \quad)$

$\vec{e}_\theta = (\quad -4\sin\varphi\sin\theta, \quad 4\sin\varphi\cos\theta, \quad 0 \quad)$

$$\vec{N} = \hat{\imath}(16\sin^2\varphi\cos\theta) - \hat{\jmath}(-16\sin^2\varphi\sin\theta) + \hat{k}(16\sin\varphi\cos\varphi\cos^2\theta + 16\sin\varphi\cos\varphi\sin^2\theta)$$

$= (16\sin^2\varphi\cos\theta, 16\sin^2\varphi\sin\theta, 16\sin\varphi\cos\varphi)$ oriented up - OK

$$\iint_H \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_H \vec{\nabla} \times \vec{F} \cdot \vec{N}\, d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2} (64\sin^2\varphi\cos\varphi\cos\theta - 64\sin^2\varphi\cos\varphi\sin\theta + 32\sin\varphi\cos\varphi)\, d\varphi\, d\theta$$

$= \int_0^{2\pi}\int_0^{\pi/2} (32\sin\varphi\cos\varphi)\, d\varphi\, d\theta$ because $\int_0^{2\pi} \cos\theta\, d\theta = \int_0^{2\pi} \sin\theta\, d\theta = 0$

$$= 2\pi\int_0^{\pi/2} (32\sin\varphi\cos\varphi)\, d\varphi = 64\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = 32\pi$$

16. (10 points) Find 3 positive numbers x, y and z, whose sum is 90 such that $f(x,y,z) = xy^2z^3$ is a maximum.
Solve either by Eliminating a Variable or by Lagrange Multipliers.
5 points extra credit for doing both. Clearly separate solutions.

METHOD 1: Lagrange Multipliers: $x + y + z = 90$

$f = xy^2z^3 \qquad \vec{\nabla}f = (y^2z^3, 2xyz^3, 3xy^2z^2) \qquad g = x + y + z \qquad \vec{\nabla}g = (1,1,1)$

$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad y^2z^3 = \lambda, \quad 2xyz^3 = \lambda, \quad 3xy^2z^2 = \lambda \quad \Rightarrow \quad y^2z^3 = 2xyz^3, \quad y^2z^3 = 3xy^2z^2$

$\Rightarrow \quad y = 2x, \quad z = 3x \quad \Rightarrow \quad x + y + z = x + 2x + 3x = 90 \quad \Rightarrow \quad 6x = 90 \quad \Rightarrow \quad x = 15$

$x = 15, \quad y = 30, \quad z = 45$

METHOD 2: Eliminate a Variable: $x + y + z = 90$

$x = 90 - y - z \quad \Rightarrow \quad f = (90 - y - z)y^2z^3 = 90y^2z^3 - y^3z^3 - y^2z^4$

$$\begin{aligned} f_y &= 180yz^3 - 3y^2z^3 - 2yz^4 = 0 \\ f_z &= 270y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 180 - 3y - 2z &= 0 \\ 270 - 3y - 4z &= 0 \end{aligned}$$

Subtract: $90 - 2z = 0 \quad \Rightarrow \quad z = 45$
Substitute back: $180 - 3y - 90 = 0 \quad \Rightarrow \quad y = 30$
Substitute back: $x = 90 - y - z = 90 - 30 - 45 = 15$