Name________________________

MATH 251 Final Exam Version B Fall 2016

Sections 504 Solutions P. Yasskin

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|---|---|---|---|
| 1-13 | /65 | 16 | /10 |
| 14 | /10 | E.C. | / 5 |
| 15 | /20 | Total | /110 |

Multiple Choice: (5 points each. No part credit.)

**1**. Find the point where the line $x = 3 - t,\ \ y = 1 + 2t,\ \ z = 1 - 2t$ intersects the plane $x - 2y + z = -2$. At this point $x + y + z =$

**a**. $0$
**b**. $1$
**c**. $2$
**d**. $3$
**e**. $4$ Correct Choice

$x - 2y + z = (3 - t) - 2(1 + 2t) + (1 - 2t) = 2 - 7t = -5 \qquad t = 1$

$x = 2,\ \ y = 3,\ \ z = -1, \qquad x + y + z = 4$

**2**. Find the plane tangent to the graph of $z = x^2y^3$ at the point $(2,1)$. The $z$-intercept is

**a**. $-24$
**b**. $-16$ Correct Choice
**c**. $-4$
**d**. $0$
**e**. $4$

$f(x,y) = x^2y^3 \qquad f_x(x,y) = 2xy^3 \qquad f_y(x,y) = 3x^2y^2$

$f(2,1) = 4 \qquad f_x(2,1) = 4 \qquad f_y(2,1) = 12$

Tan plane: $z = 4 + 4(x - 2) + 12(y - 1) = 4x + 12y - 16 \qquad z$-intercept $= -16$

**3**. Find the line perpendicular to the cone $z^2 - x^2 - y^2 = 0$ at the point $P = (4,3,5)$. This line intersects the $xy$-plane at

**a**. $(3,4,0)$
**b**. $(-4,-3,0)$
**c**. $(8,6,0)$ Correct Choice
**d**. $(-6,-8,0)$
**e**. $\left(\frac{4}{5},\frac{3}{5},0\right)$

$F = z^2 - x^2 - y^2 \qquad \vec\nabla F = (-2x,-2y,2z) \qquad \vec N = \vec\nabla F\big|_{(4,3,5)} = (-8,-6,10)$

$X = P + t\vec N = (4,3,5) + t(-8,-6,10) = (4 - 8t, 3 - 6t, 5 + 10t)$

Intersects the $xy$-plane when $z = 0$ or $5 + 10t = 0$ or $t = -\frac{1}{2}$.

So $x = 4 - 8\left(-\frac{1}{2}\right) = 8$ and $y = 3 - 6\left(-\frac{1}{2}\right) = 6$

**4**. A box currently has length $L = 20$ cm which is increasing at $4$ cm/sec, width $W = 15$ cm which is increasing at $2$ cm/sec, and height $H = 12$ cm which is decreasing at $1$ cm/sec. At what rate is the volume changing?

**a**. $3$ cm$^3$/sec
**b**. $7$ cm$^3$/sec
**c**. $540$ cm$^3$/sec
**d**. $900$ cm$^3$/sec   Correct Choice
**e**. $3600$ cm$^3$/sec

$$V = LWH \qquad \frac{dV}{dt} = WH\frac{dL}{dt} + LH\frac{dW}{dt} + LW\frac{dH}{dt} = 15 \cdot 12 \cdot 4 + 20 \cdot 12 \cdot 2 - 20 \cdot 15 \cdot 1 = 900$$

**5**. In order to hide from the Dark Invader, Duke Skywater flys the Millenium Eagle into a galactic dust storm. Currently, his position is $P = (30, -20, 10)$ and his velocity is $\vec{v} = (-4, 3, 12)$. He measures that currently the dust density is $\rho = 450$ and its gradient is $\vec{\nabla}\rho = (2, -2, 1)$. Find the current rate of change of the dust density as seen by Duke.

**a**. $-2$   Correct Choice
**b**. $2$
**c**. $110$
**d**. $448$
**e**. $560$

$$\vec{\nabla}_{\vec{v}}\rho = \vec{v} \cdot \vec{\nabla}\rho = (-4, 3, 12) \cdot (2, -2, 1) = -8 - 6 + 12 = -2$$

**6**. Under the same conditions as in #5, in what **unit** vector direction should Duke travel to **increase** the dust density as quickly as possible?

**a**. $(-2, 2, -1)$
**b**. $(2, -2, 1)$
**c**. $\left(\frac{4}{13}, \frac{-3}{13}, \frac{-12}{13}\right)$
**d**. $\left(\frac{-4}{13}, \frac{3}{13}, \frac{12}{13}\right)$
**e**. $\left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$   Correct Choice

$$\hat{u} = \frac{\vec{\nabla}\rho}{\left|\vec{\nabla}\rho\right|} = \frac{(2, -2, 1)}{\sqrt{4 + 4 + 1}} = \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)$$

7. The point $(-2,1)$ is a critical point of the function $f = \dfrac{2x - 4y - x^2y^2}{xy}$.
Use the Second Derivative Test to classify the point.

- a. Local Minimum  Correct Choice
- b. Local Maximum
- c. Inflection Point
- d. Saddle Point
- e. Test Fails

$$f = \frac{2}{y} - \frac{4}{x} - xy \qquad f_x = \frac{4}{x^2} - y \qquad f_y = -\frac{2}{y^2} - x$$

$$f_{xx} = -\frac{8}{x^3} = 1 > 0 \qquad f_{yy} = \frac{4}{y^3} = 4 > 0 \qquad f_{xy} = -1 \qquad D = 4 - 1 = 3 > 0$$ Minimum

8. Find the mass of the plate between $y = 3x$ and $y = x^2$ if the surface density is $\delta = xy$.

- a. $\frac{729}{16}$
- b. $\frac{243}{8}$  Correct Choice
- c. $\frac{81}{8}$
- d. $\frac{81}{4}$
- e. $\frac{9}{2}$

$$M = \iint \delta\, dA = \int_0^3 \int_{x^2}^{3x} xy\, dy\, dx = \int_0^3 x\left[\frac{y^2}{2}\right]_{x^2}^{3x} dx = \frac{1}{2}\int_0^3 x(9x^2 - x^4)\, dx = \frac{1}{2}\left[9\frac{x^4}{4} - \frac{x^6}{6}\right]_0^3$$
$$= \frac{1}{2}\left(\frac{3^6}{4} - \frac{3^6}{6}\right) = \frac{3^6}{2}\left(\frac{3-2}{12}\right) = \frac{3^5}{8} = \frac{243}{8}$$

9. Find the $x$-component of the center of mass of the plate between $y = 3x$ and $y = x^2$ if the surface density is $\delta = xy$.

- a. $\frac{72}{35}$  Correct Choice
- b. $\frac{35}{72}$
- c. $\frac{2187}{35}$
- d. $\frac{35}{2172}$
- e. $\frac{729}{35}$

$$M_x = \iint x\delta\, dA = \int_0^3 \int_{x^2}^{3x} x^2 y\, dy\, dx = \int_0^3 x^2\left[\frac{y^2}{2}\right]_{x^2}^{3x} dx = \frac{1}{2}\int_0^3 x^2(9x^2 - x^4)\, dx = \frac{1}{2}\left[9\frac{x^5}{5} - \frac{x^7}{7}\right]_0^3$$
$$= \frac{1}{2}\left(\frac{3^7}{5} - \frac{3^7}{7}\right) = \frac{3^7}{2}\left(\frac{7-5}{35}\right) = \frac{3^7}{35} = \frac{2187}{35}$$
$$\bar{x} = \frac{M_x}{M} = \frac{3^7}{35}\frac{8}{3^5} = \frac{72}{35}$$

**10**. Compute $\oint \vec{F} \cdot d\vec{s}$ counterclockwise around the rectangle $0 \le x \le 2\pi$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$ for $\vec{F} = (y\cos x - \sin y, x\cos y + \sin x)$.

HINT: Use the Fundamental Theorem of Calculus for Curves or Green's Theorem.

**a**. $-16\pi$
**b**. $-8\pi$
**c**. $0$
**d**. $8\pi$ Correct Choice
**e**. $16\pi$

By Green's Theorem: $\vec{F} = (P,Q)$ $\quad P = y\cos x - \sin y$ $\quad Q = x\cos y + \sin x$

$$\oint \vec{F} \cdot d\vec{s} = \iint \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy = \iint (\cos y + \cos x - \cos x + \cos y)\,dx\,dy = \int_{-\pi/2}^{\pi/2} \int_0^{2\pi} 2\cos y\,dx\,dy$$

$$= 4\pi \Big[ \sin y \Big]_{-\pi/2}^{\pi/2} = 8\pi$$

**11**. Compute $\int_{(0,0,0)}^{(2,0,1)} \vec{F} \cdot d\vec{s}$ for $\vec{F} = (2x+y, x+2y, 2z)$ along the curve $\vec{r}(t) = (t^2+t^3, t^2-t^3, \sqrt{t}\,)$.

HINT: Find a scalar potential.

**a**. $1$
**b**. $3$
**c**. $5$ Correct Choice
**d**. $7$
**e**. $9$

$\vec{\nabla} f = (\partial_x f, \partial_y f, \partial_z f) = \vec{F} = (2x+y, x+2y, 2z) \qquad \Rightarrow \qquad f = x^2 + xy + y^2 + z^2$

By the FTCC, $\int_{(0,0,0)}^{(2,0,1)} \vec{F} \cdot d\vec{s} = \int_{(0,0,0)}^{(2,0,1)} \vec{\nabla} f \cdot d\vec{s} = f(2,0,1) - f(0,0,0) = 5$

**12**. Compute $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S}$ over the sphere $x^2+y^2+z^2 = 9$ with outward normal for $\vec{F} = (xy^2z, yz^2x, zx^2y)$.

HINT: Use Stokes' Theorem or Gauss' Theorem.

**a**. $4\pi$
**b**. $12\pi$
**c**. $\frac{32}{3}\pi$
**d**. $\frac{64}{3}\pi$
**e**. $0$ Correct Choice

By Stokes' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint \vec{F} \cdot d\vec{s} = 0$ because there is no boundary curve.

By Gauss' Theorem, $\iint \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iiint \vec{\nabla} \cdot \vec{\nabla} \times \vec{F}\,dV = 0$ because $\vec{\nabla} \cdot \vec{\nabla} \times \vec{F} = 0$

**13**. Compute $\iint \vec{F} \cdot d\vec{S}$ outward through the complete surface of the cylinder $x^2 + y^2 = 4$ for $1 \le z \le 4$ for $\vec{F} = (xz^2, yz^2, z^3)$.

HINT: Use a Theorem.

**a**. $840\pi$

**b**. $420\pi$ Correct Choice

**c**. $252\pi$

**d**. $210\pi$

**e**. $126\pi$

$$\vec{\nabla} \cdot \vec{F} = z^2 + z^2 + 3z^2 = 5z^2$$

$$\iint_{\partial V} \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \int_1^4 \int_0^{2\pi} \int_0^2 5z^2\, r\, dr\, d\theta\, dz = 5\left[\frac{z^3}{3}\right]_1^4 (2\pi)\left[\frac{r^2}{2}\right]_0^2 = 5(21)(2\pi)(2) = 420\pi$$

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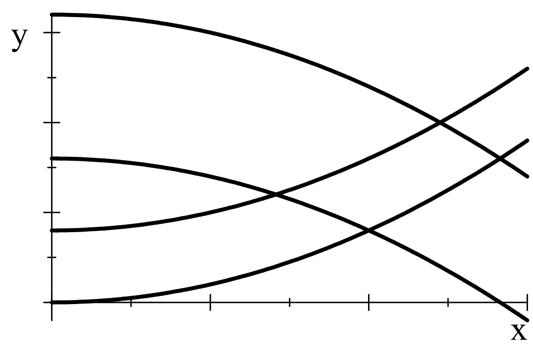
Work Out: (Points indicated. Part credit possible. Show all work.)

**14**. (10 points) Compute $\iint_D x\, dx\, dy$ over the "diamond shaped" region $D$ in the first quadrant bounded by the parabolas

$y = 16 - x^2$ $\quad$ $y = 8 - x^2$

$y = 4 + x^2$ $\quad$ and $\quad$ $y = x^2$




HINTS: Use the coordinates: $u = y + x^2$, $v = y - x^2$. Solve for $x$ and $y$.

$y = v + x^2 \qquad u = v + 2x^2 \qquad x^2 = \dfrac{u - v}{2} \qquad \boxed{x = \dfrac{\sqrt{u - v}}{\sqrt{2}}} \qquad y = v + \dfrac{u - v}{2} \qquad \boxed{y = \dfrac{u + v}{2}}$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left\|\begin{matrix} \frac{1}{\sqrt{2}}\frac{1}{2\sqrt{u-v}} & \frac{1}{2} \\ \frac{1}{\sqrt{2}}\frac{-1}{2\sqrt{u-v}} & \frac{1}{2} \end{matrix}\right\| = \left|\frac{1}{\sqrt{2}}\frac{1}{4\sqrt{u-v}} - -\frac{1}{\sqrt{2}}\frac{1}{4\sqrt{u-v}}\right| = \frac{1}{2\sqrt{2}\sqrt{u-v}}$$

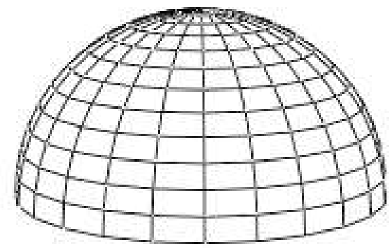
Integrand: $x = \dfrac{\sqrt{u - v}}{\sqrt{2}}$ $\qquad$ Boundaries are:

$y = 16 - x^2 \Rightarrow y + x^2 = 16 \Rightarrow u = 16 \qquad y = 8 - x^2 \Rightarrow y + x^2 = 8 \Rightarrow u = 8$

$y = 4 + x^2 \Rightarrow y - x^2 = 4 \Rightarrow v = 4 \qquad y = x^2 \Rightarrow y - x^2 = 0 \Rightarrow v = 0$

$$\iint_R x\, dx\, dy = \int_0^4 \int_8^{16} \frac{\sqrt{u-v}}{\sqrt{2}} \frac{1}{2\sqrt{2}\sqrt{u-v}}\, du\, dv = \int_0^4 \int_8^{16} \frac{1}{4}\, du\, dv = \frac{1}{4}[u]_8^{16}[v]_0^4 = \frac{1}{4}(8)(4) = 8$$

**15.** (20 points) Verify Stokes' Theorem $\iint_H \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \oint_{\partial H} \vec{F}\cdot d\vec{s}$



for the vector field $\vec{F} = (y, -x, xz + yz)$

and the hemisphere $z = \sqrt{9 - x^2 - y^2}$ oriented up.

Be sure to check the orientation. Use the following steps:

**a**. RHS: Parametrize the boundary curve and compute the line integral:

$\vec{r}(\theta) = (3\cos\theta, 3\sin\theta, 0)$

$\vec{v}(\theta) = (-3\sin\theta, 3\cos\theta, 0)$ oriented counterclockwise - OK

$\vec{F}(\vec{r}(\theta)) = (3\sin\theta, -3\cos\theta, 0)$

$$\oint_{\partial H} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}\cdot\vec{v}\,d\theta = \int_0^{2\pi} -9\sin^2\theta - 9\cos^2\theta\,d\theta = -9\int_0^{2\pi} d\theta = -18\pi$$

**b**. LHS: Compute the surface integral using the parametrization:

$\vec{R}(\varphi,\theta) = (\ 3\sin\varphi\cos\theta,\quad 3\sin\varphi\sin\theta,\quad 3\cos\varphi\ )$

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ y & -x & xz+yz \end{vmatrix} = \hat{\imath}(z) - \hat{\jmath}(z) + \hat{k}(-1-1) = (z, -z, -2)$$

$\left[\vec{\nabla}\times\vec{F}\right]_{\vec{R}(\varphi,\theta)} = (3\cos\varphi, -3\cos\varphi, -2)$

$\vec{e}_\varphi = (\ 3\cos\varphi\cos\theta,\quad 3\cos\varphi\sin\theta,\quad -3\sin\varphi\ )$

$\vec{e}_\theta = (\ -3\sin\varphi\sin\theta,\quad 3\sin\varphi\cos\theta,\quad 0\ )$

$$\vec{N} = \hat{\imath}(9\sin^2\varphi\cos\theta) - \hat{\jmath}(-9\sin^2\varphi\sin\theta) + \hat{k}(9\sin\varphi\cos\varphi\cos^2\theta + 9\sin\varphi\cos\varphi\sin^2\theta)$$
$$= (9\sin^2\varphi\cos\theta, 9\sin^2\varphi\sin\theta, 9\sin\varphi\cos\varphi)$$ oriented up - OK

$$\iint_H \vec{\nabla}\times\vec{F}\cdot d\vec{S} = \iint_H \vec{\nabla}\times\vec{F}\cdot\vec{N}\,d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2} (27\sin^2\varphi\cos\varphi\cos\theta - 27\sin^2\varphi\cos\varphi\sin\theta - 18\sin\varphi\cos\varphi)\,d\varphi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\,d\varphi\,d\theta$$ because $\int_0^{2\pi}\cos\theta\,d\theta = \int_0^{2\pi}\sin\theta\,d\theta = 0$

$$= 2\pi\int_0^{\pi/2} (-18\sin\varphi\cos\varphi)\,d\varphi = -36\pi\left[\frac{\sin^2\varphi}{2}\right]_0^{\pi/2} = -18\pi$$

**16**. (10 points) Find 3 positive numbers $x$, $y$ and $z$, whose sum is $120$ such that $f(x,y,z) = xy^2z^3$ is a maximum.

Solve either by Eliminating a Variable or by Lagrange Multipliers.

5 points extra credit for doing both. Clearly separate solutions.

METHOD 1: Lagrange Multipliers: $x+y+z = 120$

$f = xy^2z^3 \quad \vec{\nabla}f = (y^2z^3, 2xyz^3, 3xy^2z^2) \quad g = x+y+z \quad \vec{\nabla}g = (1,1,1)$

$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad y^2z^3 = \lambda, \quad 2xyz^3 = \lambda, \quad 3xy^2z^2 = \lambda \quad \Rightarrow \quad y^2z^3 = 2xyz^3, \quad y^2z^3 = 3xy^2z^2$

$\Rightarrow \quad y = 2x, \quad z = 3x \quad \Rightarrow \quad x+y+z = x+2x+3x = 120 \quad \Rightarrow \quad 6x = 120 \quad \Rightarrow \quad x = 20$

$x = 20, \quad y = 40, \quad z = 60$

METHOD 2: Eliminate a Variable: $x+y+z = 120$

$x = 120-y-z \quad \Rightarrow \quad f = (120-y-z)y^2z^3 = 120y^2z^3 - y^3z^3 - y^2z^4$

$$\begin{array}{lcl} f_y = 240yz^3 - 3y^2z^3 - 2yz^4 = 0 & \Rightarrow & 240 - 3y - 2z = 0 \\ f_z = 360y^2z^2 - 3y^3z^2 - 4y^2z^3 = 0 & & 360 - 3y - 4z = 0 \end{array}$$

Subtract: $120 - 2z = 0 \quad \Rightarrow \quad z = 60$

Substitute back: $240 - 3y - 120 = 0 \quad \Rightarrow \quad y = 40$

Substitute back: $x = 120 - y - z = 120 - 40 - 60 = 20$