Name____________________ ID__________ Section_____

1-8	/40	11	/15
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MATH 253 Exam 2 Fall 2003

Sections 504-506 Solutions P. Yasskin

Multiple Choice: (5 points each) Work Out: (15 points each)

1. Compute $\int_1^2 \int_0^3 e^{x+y}\,dx\,dy$.

a. $e^5 - e^4$

b. $e^5 - e^2 - e^4 + e$ correctchoice

c. $e^5 - e^2 - e^4 - e$

d. $e^5 + e^2 - e^4 - e$

e. $e^5 + e^4$

$$\int_1^2 \int_0^3 e^{x+y}\,dx\,dy = \int_1^2 \left[e^{x+y}\right]_0^3 dy = \int_1^2 (e^{3+y} - e^y)\,dy = \left[e^{3+y} - e^y\right]_1^2$$
$$= (e^5 - e^2) - (e^4 - e^1) = e^5 - e^2 - e^4 + e$$

2. Compute $\int_1^2 \int_{-x}^x y\,dy\,dx$.

a. 0 correctchoice

b. $\frac{7}{6}$

c. $\frac{4}{3}$

d. $\frac{7}{3}$

e. $\frac{8}{3}$

$$\int_1^2 \int_{-x}^x y\,dy\,dx = \int_1^2 \left[\frac{y^2}{2}\right]_{-x}^x dx = \int_1^2 0\,dx = 0$$

3. Rewrite the polar equation $r^2 = \sin 2\theta$ in rectangular coordinates.

a. $x^4 + y^4 = 2xy$

b. $(x^2 + y^2)^2 = 2xy$ correctchoice

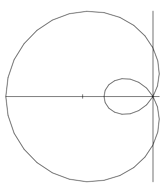
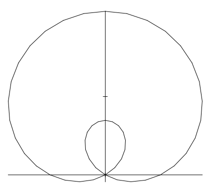
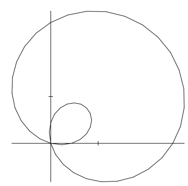
c. $(x^2 + y^2)^{3/2} = 2y$

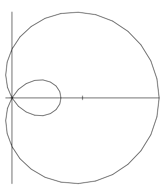
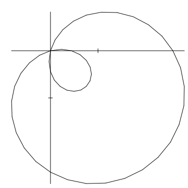
d. $(x^2 + y^2)^{3/2} = 2x$

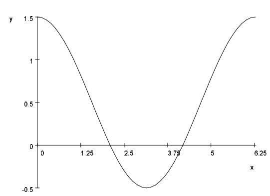
e. $x^3 + y^3 = 2y$

$$r^2 = \sin 2\theta = 2\sin\theta\cos\theta = 2\frac{y}{r}\frac{x}{r} \quad\Rightarrow\quad r^4 = 2xy \quad\Rightarrow\quad (x^2+y^2)^2 = 2xy$$

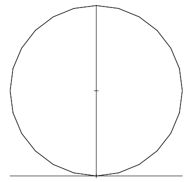
4. Which of the following is the graph of the polar equation $r = \frac{1}{2} + \cos\theta$?

a.  **b.**  c. 

d.  correctchoice **e.** 

The rectangular graph is 

. So when $\theta = 0, r = 1.5,$ etc.

5. The graph of $r = 4\sin\theta$ is  . Find the area enclosed.

HINT: What is the interval for θ?

a. π

b. 2π

c. 4π correctchoice

d. 8π

e. 16π

$$A = \iint 1\,dA = \int_0^{\pi}\int_0^{4\sin\theta} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{4\sin\theta} d\theta = \int_0^{\pi} \frac{16\sin^2\theta}{2}\,d\theta = 8\int_0^{\pi} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= 4\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi} = 4\pi$$

6. Compute $\int_0^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)^3\,dy\,dx$.

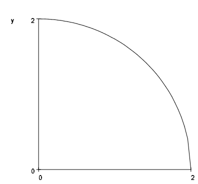
a. π

b. 2π

c. 4π

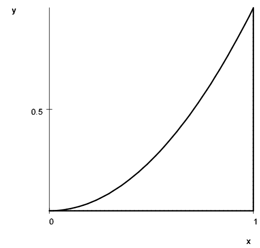
d. 8π

e. 16π correctchoice

The region is  . Switch to polar:

$$\int_0^2\int_0^{\sqrt{4-x^2}} (x^2+y^2)^3\,dy\,dx = \int_0^{\pi/2}\int_0^2 (r^2)^3\,r\,dr\,d\theta = \frac{\pi}{2}\int_0^2 r^7\,dr = \frac{\pi}{2}\,\frac{r^8}{8}\Big|_0^2 = 16\pi$$

7. Compute $\iint_D x\cos y\,dA$ over the region D bounded by $y = 0$, $y = x^2$ and $x = 1$.



a. $\frac{-\cos 1}{2}$

b. $\sin 1$

c. $\frac{\sin 1}{2}$

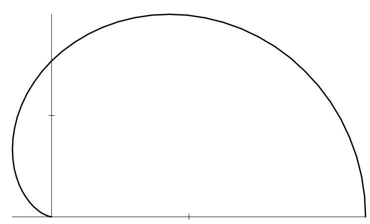
d. $\frac{1}{2}(1 - \sin 1)$

e. $\frac{1}{2}(1 - \cos 1)$ correctchoice

$$\iint_D x\cos y\,dA = \int_0^1 \int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \Big[x\sin y\Big]_0^{x^2} dx = \int_0^1 x\sin(x^2)\,dx = \frac{-\cos(x^2)}{2}\Bigg|_0^1$$

$$= \frac{-\cos 1}{2} - \frac{-\cos 0}{2} = \frac{1}{2}(1 - \cos 1)$$

8. A styrofoam board is cut in the shape of the upper half of the cardioid $r = 1 + \cos\theta$.
A static electricity charge is put on the board whose surface charge density is given by $\rho_e = y$.
Find the total charge on the board $Q = \iint \rho_e\,dA$.



a. 0

b. $\frac{2}{3}$

c. $\frac{4}{3}$ correctchoice

d. $\frac{8\pi}{3}$

e. $\frac{16\pi}{3}$

$$Q = \iint \rho_e\,dA = \int_0^{\pi} \int_0^{1+\cos\theta} r\sin\theta\, r\,dr\,d\theta = \int_0^{\pi} \left[\frac{r^3}{3}\sin\theta\right]_0^{1+\cos\theta} d\theta = \int_0^{\pi} \frac{(1+\cos\theta)^3}{3}\sin\theta\,d\theta$$

$$= \left[\frac{-(1+\cos\theta)^4}{12}\right]_0^{\pi} = \frac{-0}{12} - \frac{-2^4}{12} = \frac{4}{3}$$

9. Find all critical points of the function $f(x,y) = 2x^3 + 3x^2y + y^3 - 12y$ and classify each as a local maximum, a local minimum or a saddle point. (Make a table.)

$f_x = 6x^2 + 6xy = 6x(x+y) = 0 \qquad f_y = 3x^2 + 3y^2 - 12 = 0$

Case: $x = 0$: Then $3y^2 - 12 = 0$ or $y = \pm 2$. Crit. pts: $(0,2)$ and $(0,-2)$

Case: $x = -y$: Then $6y^2 - 12 = 0$ or $y = \pm\sqrt{2}$. Crit. pts: $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$

x	y	$f_{xx} = 12x + 6y$	$f_{yy} = 6y$	$f_{xy} = 6x$	$D = f_{xx}f_{yy} - f_{xy}^2$	Classification
0	2	12	12	0	144	local minimum
0	-2	-12	-12	0	144	local maximum
$\sqrt{2}$	$-\sqrt{2}$	$6\sqrt{2}$	$-6\sqrt{2}$	$6\sqrt{2}$	-144	saddle point
$-\sqrt{2}$	$\sqrt{2}$	$-6\sqrt{2}$	$6\sqrt{2}$	$-6\sqrt{2}$	-144	saddle point

10. Consider a box (like a donut box) whose lid folds closed so that when closed there are two layers of cardboard in the front and on each of the two sides while there is only one layer of cardboard on the top, bottom and back. If the box holds 3 m^3, what are the dimensions which use the least amount of cardboard? Let L be the length side to side. Let W be the width front to back and let H be the height.

We minimize the surface area: $A = 2LW + 3LH + 4WH$

subject to the volume constraint: $V = LWH = 3$

METHOD 1: $H = \dfrac{3}{LW} \qquad A = 2LW + \dfrac{9}{W} + \dfrac{12}{L}$

$A_L = 2W - \dfrac{12}{L^2} = 0 \qquad A_W = 2L - \dfrac{9}{W^2} = 0$

So $W = \dfrac{6}{L^2}$ and $2L = \dfrac{9}{W^2} = \dfrac{9L^4}{36} = \dfrac{L^4}{4}$ So $L^3 = 8$

Therefore $L = 2 \qquad W = \dfrac{6}{4} = \dfrac{3}{2} \qquad H = \dfrac{3 \cdot 2}{2 \cdot 3} = 1$

METHOD 2: $\vec{\nabla} A = (2W + 3H, 2L + 4H, 3L + 4W) \qquad \vec{\nabla} V = (WH, LH, LW)$

$\vec{\nabla} A = \lambda \vec{\nabla} V$: $\quad 2W + 3H = \lambda WH \qquad 2L + 4H = \lambda LH \qquad 3L + 4W = \lambda LW$

Solve each for λ: $\lambda = \dfrac{2}{H} + \dfrac{3}{W} = \dfrac{2}{H} + \dfrac{4}{L} = \dfrac{3}{W} + \dfrac{4}{L}$

$\dfrac{3}{W} = \dfrac{4}{L} \Rightarrow W = \dfrac{3}{4}L \qquad \dfrac{2}{H} = \dfrac{4}{L} \Rightarrow H = \dfrac{1}{2}L$

$LWH = 3 \Rightarrow L\left(\frac{3}{4}L\right)\left(\frac{1}{2}L\right) = 3 \Rightarrow L^3 = 8 \Rightarrow L = 2 \Rightarrow W = \frac{3}{2} \qquad H = 1$

11. Find the mass and center of mass of the region between the parabola $y = x^2$ and the line $y = 4$, if the surface density is given by $\rho = y$.

$$M = \iint \rho\, dA = \int_{-2}^{2}\int_{x^2}^{4} y\, dy\, dx = \int_{-2}^{2}\left[\frac{y^2}{2}\right]_{y=x^2}^{4} dx = \int_{-2}^{2}\left[8 - \frac{x^4}{2}\right] dx = \left[8x - \frac{x^5}{10}\right]_{-2}^{2}$$

$$= 2\left[16 - \frac{32}{10}\right] = 32\left[1 - \frac{1}{5}\right] = \frac{128}{5}$$

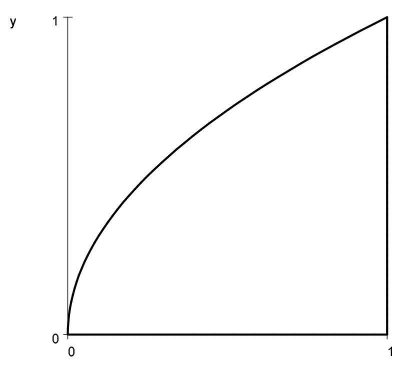
By symmetry $\bar{x} = 0$.

$$y\text{-mom} = \iint y\rho\, dA = \int_{-2}^{2}\int_{x^2}^{4} y^2\, dy\, dx = \int_{-2}^{2}\left[\frac{y^3}{3}\right]_{y=x^2}^{4} dx = \int_{-2}^{2}\left[\frac{64}{3} - \frac{x^6}{3}\right] dx = \left[\frac{64}{3}x - \frac{x^7}{21}\right]_{-2}^{2}$$

$$= 2\left[\frac{128}{3} - \frac{128}{21}\right] = 256\left[\frac{7}{21} - \frac{1}{21}\right] = \frac{512}{7}$$

$$\bar{y} = \frac{y\text{-mom}}{M} = \frac{512}{7} \cdot \frac{5}{128} = \frac{20}{7} \approx 2.9$$

12. Sketch the region of integration and then compute the integral $\int_0^1\int_{y^2}^1 y^3 \sin(x^3)\, dx\, dy$.



Reverse the order of integration:

$$\int_0^1\int_{y^2}^1 y^3 \sin(x^3)\, dx\, dy = \int_0^1\int_0^{\sqrt{x}} y^3 \sin(x^3)\, dy\, dx = \int_0^1\left[\frac{y^4}{4}\sin(x^3)\right]_{y=0}^{\sqrt{x}} dx = \int_0^1 \frac{x^2}{4}\sin(x^3)\, dx$$

Substitute: $u = x^3$ $\quad du = 3x^2\, dx$ $\quad x^2\, dx = \frac{1}{3}du$

$$\int_0^1\int_{y^2}^1 y^3 \sin(x^3)\, dx\, dy = \frac{1}{12}\int \sin(u)\, du = \frac{-1}{12}\cos(u) = \frac{-1}{12}\cos(x^3)\Big|_0^1$$

$$= \frac{-1}{12}[\cos 1 - \cos 0] = \frac{1}{12}(1 - \cos 1)$$