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MATH 253 Exam 2 Spring 2004

Sections 504-506 Solutions P. Yasskin

On the front of the Blue Book, on the Scantron and on this sheet
write your Name, your University ID, your Section and "Exam 2."
On the front of the Blue Book copy the Grading Grid shown at the right.
Enter your Multiple Choice answers on the Scantron
and CIRCLE them on this sheet.

Multiple Choice: (6 points each. No part credit.)

1. Find the equation of the plane tangent to the graph of the function $f(x,y) = x^2y$ at the point $(3,2,18)$.

a. $z = 12x + 9y + 18$

b. $z = 9x + 12y + 18$

c. $z = 12x + 9y - 36$ correctchoice

d. $z = 9x + 12y - 18$

e. $z = 4x^2y - 6xy - 2x^2$

$f_x(x,y) = 2xy \quad f_y(x,y) = x^2 \quad f(3,2) = 18 \quad f_x(3,2) = 12 \quad f_y(3,2) = 9$
$z = f(3,2) + f_x(3,2)(x-3) + f_y(3,2)(y-2) = 18 + 12(x-3) + 9(y-2) = 12x + 9y - 36$

2. The equation of the plane tangent to the graph of $z = f(x,y)$ at $(1,2)$ is $z = 4 + 2(x-1) - 3(y-2)$. Use the linear approximation to estimate $f(1.2, 1.9)$.

a. 0.7

b. 3.3

c. 3.9

d. 4.1

e. 4.7 correctchoice

$f(1.2,1.9) \approx 4 + 2(1.2-1) - 3(1.9-2) = 4 + 2(.2) - 3(-.1) = 4.7$

3. Find the equation of the plane tangent to the ellipsoid $\frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} = 1$ at the point $(1,2,3)$.

a. $6x + 3y + 2z = 18$ correctchoice

b. $6x + 3y + 2z = -18$

c. $36x + 9y + 4z = 66$

d. $36x + 9y + 4z = -66$

e. $3x + 2y + z = 10$

$f(x,y,z) = \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} \qquad \vec{\nabla} f = \left(\frac{2x}{3}, \frac{y}{6}, \frac{2z}{27}\right) \qquad P = (1,2,3) \qquad \vec{N} = \vec{\nabla} f\Big|_{(1,2,3)} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{9}\right)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad \frac{2}{3}x + \frac{1}{3}y + \frac{2}{9}z = \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 2 + \frac{2}{9} \cdot 3 = \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 2 \qquad 6x + 3y + 2z = 18$

4. If the temperature is $T(x,y,z) = \frac{xy}{z}$ and a bird is at $(x,y,z) = (3,2,1)$, in what unit vector direction should the bird fly to warm up as quick as possible?

a. $(3,2,-6)$

b. $\left(\frac{3}{7}, \frac{2}{7}, \frac{-6}{7}\right)$

c. $(2,3,-6)$

d. $\left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$ correctchoice

e. $(3,2,6)$

$\vec{\nabla} T = \left(\frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2}\right) = (2,3,-6) \qquad \left|\vec{\nabla} T\right| = \sqrt{4+9+36} = \sqrt{49} = 7 \qquad \frac{\vec{\nabla} T}{\left|\vec{\nabla} T\right|} = \left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$

5. If the temperature is $T(x,y,z) = \frac{xy}{z}$ and a bird is at $(x,y,z) = (3,2,1)$ flying with velocity $\vec{v} = (2,1,3)$, what is the rate of change of the temperature as seen by the bird?

a. -25

b. -11 correctchoice

c. 0

d. 11

e. 25

$\vec{\nabla} T = \left(\frac{y}{z}, \frac{x}{z}, \frac{-xy}{z^2}\right) = (2,3,-6) \qquad \nabla_{\vec{v}} T = \vec{v} \cdot \vec{\nabla} T = (2,1,3) \cdot (2,3,-6) = -11$

6. If $w = x^3 + y^3$ where $x = \cos(pq)$ and $y = \sin(pq)$, find $\frac{\partial w}{\partial q}$ at $(p,q) = \left(\frac{1}{6}, \pi\right)$.

NOTE: $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$ $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

a. $\frac{\sqrt{3}-3}{16}$ correctchoice

b. $\frac{3-\sqrt{3}}{16}$

c. $\frac{\sqrt{3}+3}{16}$

d. $\frac{-3-\sqrt{3}}{16}$

e. $\frac{3-2\sqrt{3}}{16}$

METHOD 1: 2 variable Chain Rule:

$$\frac{\partial w}{\partial q} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial q} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial q} = (3x^2)[-\sin(pq)p] + (3y^2)[\cos(pq)p]$$

$$= -3p\cos^2(pq)\sin(pq) + 3p\sin^2(pq)\cos(pq)$$

$$= -3 \cdot \frac{1}{6}\cos^2\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + 3 \cdot \frac{1}{6}\sin^2\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) = -\frac{1}{2}\frac{3}{4}\frac{1}{2} + \frac{1}{2}\frac{1}{4}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}-3}{16}$$

METHOD 2: Find the composition and use the 1 variable chain rule:

$$w = \cos^3(pq) + \sin^3(pq) \qquad \frac{\partial w}{\partial q} = 3\cos^2(pq)[-\sin(pq)p] + 3\sin^2(pq)[\cos(pq)p] = \cdots = \frac{\sqrt{3}-3}{16}$$

7. Suppose $f(x,y) = x^2 y$ where $x = x(u,v)$ and $y = y(u,v)$. Find $\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(3,4)}$ given that:

$$x(3,4) = 1 \qquad \left.\frac{\partial x}{\partial u}\right|_{(u,v)=(3,4)} = 5 \qquad \left.\frac{\partial x}{\partial v}\right|_{(u,v)=(3,4)} = 7$$

$$y(3,4) = 2 \qquad \left.\frac{\partial y}{\partial u}\right|_{(u,v)=(3,4)} = 6 \qquad \left.\frac{\partial y}{\partial v}\right|_{(u,v)=(3,4)} = 8$$

a. 13

b. 26 correctchoice

c. 52

d. 83

e. 174

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,2)} = 2xy|_{(x,y)=(1,2)} = 4 \qquad \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,2)} = x^2|_{(x,y)=(1,2)} = 1$$

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(3,4)} = \left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,2)}\left.\frac{\partial x}{\partial u}\right|_{(u,v)=(3,4)} + \left.\frac{\partial f}{\partial y}\right|_{(x,y)=(1,2)}\left.\frac{\partial y}{\partial u}\right|_{(u,v)=(3,4)} = 4 \cdot 5 + 1 \cdot 6 = 26$$

Work Out: (15 points each. Part credit possible.)

Start each problem on a new page of the Blue Book. Number the problem. Show all work.

8. $\left(15 \text{ points}\right)$ Find 3 positive numbers a, b and c, whose product is 36 for which $a+2b+3c$ is a minimum.
You MUST solve the problem by Eliminating a Variable.

Minimize $f = a+2b+3c$ subject to the constraint $g = abc = 36$.

$a = \dfrac{36}{bc}$ $\qquad f = \dfrac{36}{bc} + 2b + 3c$ $\qquad f_b = \dfrac{-36}{b^2c} + 2 = 0$ and $f_c = \dfrac{-36}{bc^2} + 3 = 0$

From f_b: $\quad 2 = \dfrac{36}{b^2c} \quad \Rightarrow \quad c = \dfrac{18}{b^2}$

From f_c: $\quad 3bc^2 = 36 \quad \Rightarrow \quad 3b\left(\dfrac{18}{b^2}\right)^2 = 36 \quad \Rightarrow \quad \dfrac{3 \cdot 18 \cdot 18}{b^3} = 36$

$\Rightarrow \quad b^3 = \dfrac{3 \cdot 18 \cdot 18}{36} = 27 \quad \Rightarrow \quad b = 3 \qquad c = \dfrac{18}{b^2} = \dfrac{18}{3^2} = 2 \qquad a = \dfrac{36}{bc} = \dfrac{36}{3 \cdot 2} = 6$

Solution: $\quad a = 6, \quad b = 3, \quad c = 2$

9. $\left(15 \text{ points}\right)$ Find 3 positive numbers a, b and c, whose product is 36 for which $a+2b+3c$ is a minimum.
You MUST solve the problem by the Method of Lagrange Multipliers.

Minimize $f = a+2y+3z$ subject to the constraint $g = abc = 36$.

$\vec{\nabla} f = (1,2,3) \qquad \vec{\nabla} g = (bc, ac, ab)$

Lagrange equations: $\quad \vec{\nabla} f = \lambda \vec{\nabla} g$: $\quad 1 = \lambda bc \qquad 2 = \lambda ac \qquad 3 = \lambda ab$

$\lambda = \dfrac{1}{bc} = \dfrac{2}{ac} = \dfrac{3}{ab} \quad \Rightarrow \quad a = 2b$ and $c = \dfrac{2b}{3}$

$abc = 36 \quad \Rightarrow \quad (2b)b\left(\dfrac{2b}{3}\right) = 36 \quad \Rightarrow \quad b^3 = \dfrac{3 \cdot 36}{4} = 27$

$b = 3 \qquad c = \dfrac{2b}{3} = \dfrac{2 \cdot 3}{3} = 2 \qquad a = 2b = 2 \cdot 3 = 6$

Solution: $\quad a = 6, \quad b = 3, \quad c = 2$

10. (15 points) Suppose x, y and z are related by the equation $yz + xz + xy = 11$.

a. Use implicit differentiation to compute $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1,2,3)$.

Apply $\frac{\partial}{\partial x}$ to both sides: (Remember, y is constant and z is a function of x and y.)

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial x} + z + y = 0 \quad \Rightarrow \quad (y+x)\frac{\partial z}{\partial x} = -z - y \quad \Rightarrow \quad \frac{\partial z}{\partial x} = \frac{-z-y}{y+x} \quad \Rightarrow \quad \left.\frac{\partial z}{\partial x}\right|_{(1,2,3)} = \frac{-5}{3}$$

Apply $\frac{\partial}{\partial y}$ to both sides: (Remember, x is constant and z is a function of x and y.)

$$y\frac{\partial z}{\partial y} + z + x\frac{\partial z}{\partial y} + x = 0 \quad \Rightarrow \quad (y+x)\frac{\partial z}{\partial y} = -z - x \quad \Rightarrow \quad \frac{\partial z}{\partial y} = \frac{-z-x}{y+x} \quad \Rightarrow \quad \left.\frac{\partial z}{\partial y}\right|_{(1,2,3)} = \frac{-4}{3}$$

b. Find the equation of the plane tangent to the surface $yz + xz + xy = 11$ at the point $(1,2,3)$.

METHOD 1: The surface implicitly defines $z = f(x,y)$ and we need the tangent plane at $(a,b) = (1,2)$ with $f(1,2) = 3$.

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = 3 + \frac{-5}{3}(x-1) + \frac{-4}{3}(y-2) = -\frac{5}{3}x - \frac{4}{3}y + \frac{22}{3}$$

So the tangent plane is $5x + 4y + 3z = 22$.

METHOD 2: Let $F = yz + xz + xy$. The tangent plane is $\vec{N} \cdot X = \vec{N} \cdot P$

$$P = (1,2,3) \qquad \vec{\nabla} F = (z+y, z+x, y+x) \qquad \vec{N} = \left.\vec{\nabla} F\right|_{(1,2,3)} = (5,4,3)$$

So the tangent plane is $5x + 4y + 3z = 5 \cdot 1 + 4 \cdot 2 + 3 \cdot 3 = 22$

11. (15 points) Find all critical points of the function $f(x,y) = 8x^3 + y^3 - 12xy$ and classify each as a local minimum, a local maximum or a saddle point. Be sure to say why.

NOTE: $A^3 - B^3 = (A-B)(A^2 + AB + B^2)$

$f_x = 24x^2 - 12y = 0 \qquad f_y = 3y^2 - 12x = 0$

From f_x: $y = 2x^2$ From f_y: $4x = y^2 = (2x^2)^2 = 4x^4$

So $x = x^4$ or $x^4 - x = 0$ or $x(x^3 - 1) = 0$ or $x(x-1)(x^2 + x + 1) = 0$. So either $x = 0$ or $x = 1$.

If $x = 0$, then $y = 2x^2 = 0$. If $x = 1$, then $y = 2x^2 = 2$.

So the critical points are $(0,0)$ and $(1,2)$.

Apply the Second Derivative Test:

$f_{xx} = 48x \qquad f_{yy} = 6y \qquad f_{xy} = -12 \qquad D = f_{xx}f_{yy} - f_{xy}^2 = 288xy - 144$

$f_{xx}(0,0) = 0 \qquad D(0,0) = -144 < 0$ So $(0,0)$ is a saddle point.

$f_{xx}(1,2) = 48 > 0 \qquad D(1,2) = 288 \cdot 2 - 144 > 0$ So $(1,2)$ is a local minimum.