

Line & Surface Integral Notation

PARAMETRIZED CURVES & LINE INTEGRALS:

Curve:

$$\vec{r}(t) = (x(t), y(t), z(t))$$

Tangent vector:

$$\vec{v} = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right)$$

Tangent differential vector:

$$d\vec{s} = d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k} = \left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} \right) dt = \vec{v} dt = \hat{v}|\vec{v}| dt = \hat{v} ds$$

Tangent differential scalar:

$$ds = |ds| = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = |\vec{v}| dt$$

Arclength integral:

$$s = \int_A^B ds = \int_a^b |\vec{v}| dt$$

Integral of a scalar $f(x, y, z)$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B f ds = \int_a^b f(\vec{r}(t)) |\vec{v}| dt$$

Total mass:

$$M = \int_A^B \rho ds = \int_a^b \rho |\vec{v}| dt$$

Center of mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M} \int_A^B (x, y, z) \rho ds$$

Integral of a vector field $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ along $\vec{r}(t)$ from $A = \vec{r}(a)$ to $B = \vec{r}(b)$:

$$\int_A^B \vec{F} \cdot d\vec{s} = \int_A^B (F_1 dx + F_2 dy + F_3 dz) = \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt = \int_a^b \vec{F} \cdot \vec{v} dt = \int_A^B \vec{F} \cdot \hat{v} ds$$

SPECIAL FOR CURVES IN R^2 :

Normal vector:

$$\vec{n} = \vec{v}^\perp = \begin{vmatrix} \hat{i} & \hat{j} \\ v_1 & v_2 \end{vmatrix} = v_2\hat{i} - v_1\hat{j} = \frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j} \quad |\vec{n}| = |\vec{v}|$$

Normal differential vector:

$$d\vec{n} = dy\hat{i} - dx\hat{j} = \left(\frac{dy}{dt}\hat{i} - \frac{dx}{dt}\hat{j} \right) dt = \vec{n} dt = \hat{n}|\vec{n}| dt = \hat{n} ds$$

Normal differential scalar:

$$dn = |d\vec{n}| = \sqrt{(dy)^2 + (dx)^2} = ds$$

Integral of the normal component of vector field $\vec{G} = G_1\hat{i} + G_2\hat{j}$ along $\vec{r}(t)$:

$$\int_A^B \vec{G} \cdot d\vec{n} = \int_A^B (G_1 dy - G_2 dx) = \int_a^b \left(G_1 \frac{dy}{dt} - G_2 \frac{dx}{dt} \right) dt = \int_a^b \vec{G} \cdot \vec{n} dt = \int_A^B \vec{G} \cdot \hat{n} ds$$

Further, if $\vec{G} = \vec{F}^\perp = F_2\hat{i} - F_1\hat{j}$ then:

$$\vec{G} \cdot \vec{n} = (F_2, -F_1) \cdot (v_2, -v_1) = \vec{F} \cdot \vec{v}$$

and

$$\int_A^B \vec{G} \cdot d\vec{n} = \int_A^B (F_2 dy - (-F_1) dx) = \int_A^B \vec{F} \cdot d\vec{s}$$

PARAMETRIZED SURFACES & SURFACE INTEGRALS:

Surface:

$$\vec{R}(u, v) = (x(u, v)y(u, v)z(u, v))$$

Tangent vectors:

$$\vec{e}_u = \frac{\partial \vec{R}}{\partial u} = \frac{\partial x}{\partial u} \hat{i} + \frac{\partial y}{\partial u} \hat{j} + \frac{\partial z}{\partial u} \hat{k} \quad \text{and} \quad \vec{e}_v = \frac{\partial \vec{R}}{\partial v} = \frac{\partial x}{\partial v} \hat{i} + \frac{\partial y}{\partial v} \hat{j} + \frac{\partial z}{\partial v} \hat{k}$$

Normal vector:

$$\vec{N} = \vec{e}_u \times \vec{e}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} = \frac{\partial(y, z)}{\partial(u, v)} \hat{i} + \frac{\partial(z, x)}{\partial(u, v)} \hat{j} + \frac{\partial(x, y)}{\partial(u, v)} \hat{k}$$

Surface differential vector:

$$\begin{aligned} \vec{dS} &= dy dz \hat{i} + dz dx \hat{j} + dx dy \hat{k} = \left(\frac{\partial(y, z)}{\partial(u, v)} \hat{i} + \frac{\partial(z, x)}{\partial(u, v)} \hat{j} + \frac{\partial(x, y)}{\partial(u, v)} \hat{k} \right) du dv \\ &= \vec{N} du dv = \hat{N} |\vec{N}| du dv = \hat{N} dS \end{aligned}$$

Surface differential scalar:

$$\begin{aligned} dS &= |\vec{dS}| = \sqrt{(dy dz)^2 + (dz dx)^2 + (dx dy)^2} = \sqrt{\left(\frac{\partial(y, z)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(z, x)}{\partial(u, v)} \right)^2 + \left(\frac{\partial(x, y)}{\partial(u, v)} \right)^2} du dv \\ &= |\vec{N}| du dv \end{aligned}$$

Surface area integral:

$$A = \iint_{\vec{R}} dS = \iint_{\vec{R}} |\vec{N}| du dv$$

Integral of a scalar $f(x, y, z)$ over $\vec{R}(u, v)$:

$$\iint_{\vec{R}} f dS = \iint_{\vec{R}} f(\vec{R}(u, v)) |\vec{N}| du dv$$

Total mass:

$$M = \iint_{\vec{R}} \rho dS = \iint_{\vec{R}} \rho |\vec{N}| du dv$$

Center of mass:

$$(\bar{x}, \bar{y}, \bar{z}) = \frac{1}{M} \iint_{\vec{R}} (x, y, z) \rho dS$$

Integral of a vector field $\vec{F} = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$ over $\vec{R}(u, v)$:

$$\begin{aligned} \iint_{\vec{R}} \vec{F} \cdot \vec{dS} &= \iint_{\vec{R}} (F_1 dy dz + F_2 dz dx + F_3 dx dy) = \iint_{\vec{R}} \left(F_1 \frac{\partial(y, z)}{\partial(u, v)} + F_2 \frac{\partial(z, x)}{\partial(u, v)} + F_3 \frac{\partial(x, y)}{\partial(u, v)} \right) du dv \\ &= \iint_{\vec{R}} \vec{F} \cdot \vec{N} du dv = \iint_{\vec{R}} \vec{F} \cdot \hat{N} dS \end{aligned}$$