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MATH 253 Exam 1 Spring 2007

Sections 501-503 Solutions P. Yasskin

Multiple Choice: (5 points each. No part credit.)

1. Find the area of the triangle whose vertices are
$P = (3,4,-5)$, $Q = (3,5,-4)$ and $R = (5,2,-5)$.

a. $\sqrt{3}$ Correct Choice

b. $2\sqrt{3}$

c. $4\sqrt{3}$

d. 1

e. 6

$\overrightarrow{PQ} = Q - P = \langle 0,1,1\rangle \qquad \overrightarrow{PR} = R - P = \langle 2,-2,0\rangle$

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 2 & -2 & 0 \end{vmatrix} = \hat{\imath}(0--2) - \hat{\jmath}(0-2) + \hat{k}(0-2) = \langle 2,2,-2\rangle$$

$$A = \frac{1}{2}\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = \frac{1}{2}\sqrt{4+4+4} = \sqrt{3}$$

2. Which of the following is a line perpendicular to the plane $2x - 3y + z = 1$?

a. $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{1}$

b. $\dfrac{x-2}{1} = \dfrac{y-3}{2} = \dfrac{z-1}{3}$

c. $2x + 3y + z = -1$

d. $(x,y,z) = (1+2t, 2+3t, 3+t)$

e. $(x,y,z) = (1+2t, 2-3t, 3+t)$ Correct Choice

The normal to the plane is $\vec{N} = \langle 2,-3,1\rangle$, which must be the direction $\vec{v} = \langle v_1,v_2,v_3\rangle = \langle 2,-3,1\rangle$ of the line in either the parametric form $X = P + t\vec{v}$ or the symmetric form $\dfrac{x-p}{v_1} = \dfrac{y-q}{v_2} = \dfrac{z-r}{v_3}$.

3. An airplane is travelling due North with constant speed and constant altitude as it flies over College Station. Since its path is part of a circle around the earth, its acceleration points directly toward the center of the earth. In which direction does it binormal $\hat{B}$ point?

a. North

b. East

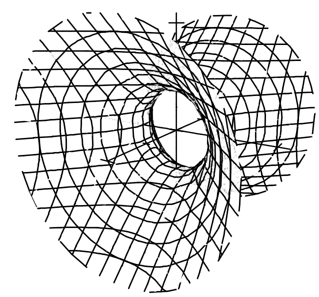
c. South

d. West   Correct Choice

e. Up

$\vec{v}$ is North. $\vec{a}$ is Down. So $\hat{B} = \dfrac{\hat{v} \times \hat{a}}{|\hat{v} \times \hat{a}|}$ points West by the right hand rule.

4. The plot at the right is which surface?



a. $x^2 - y^2 - z^2 = 4$

b. $x^2 - y^2 - z^2 = -4$ Correct Choice

c. $4x^2 + y^2 + z^2 = 1$

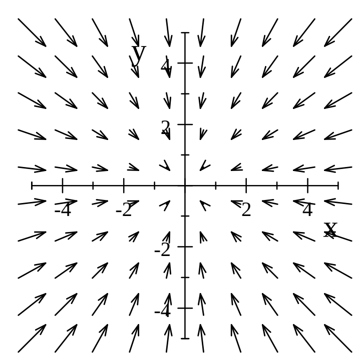
d. $x = 4y^2 - 4z^2$

e. $x = 4y^2 + 4z^2$

This is a hyperboloid of 1 sheet.   c is an ellipsoid.   d and e are paraboloids.

b is correct because the equation $x^2 + 4 = y^2 + z^2$ shows $y^2 + z^2 \geq 4$.

5. The plot at the right represents which vector field?



a. $\vec{A} = \langle x, y \rangle$

b. $\vec{B} = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$

c. $\vec{C} = \langle -x, -y \rangle$ Correct Choice

d. $\vec{D} = \left\langle \dfrac{-x}{\sqrt{x^2+y^2}}, \dfrac{-y}{\sqrt{x^2+y^2}} \right\rangle$

e. $\vec{E} = \langle -y, x \rangle$

The vectors all point radially inward. So it must be either: $\vec{C}$ or $\vec{D}$.

The vectors get shorter near the origin. So it cannot be $\vec{D}$ which is a unit vector field.

6. For the curve $\vec{r}(t) = \left(e^t, \quad \sqrt{2}\,t, \quad e^{-t}\right)$ which of the following is FALSE?

a. $\vec{v} = \left\langle e^t, \quad \sqrt{2}, \quad -e^{-t}\right\rangle$

b. $|\vec{v}| = e^t + e^{-t}$

c. Arc length between $t = 0$ and $t = 1$ is $e + \frac{1}{e}$ Correct Choice

d. $\vec{a} = \langle e^t, \quad 0, \quad e^{-t}\rangle$

e. $a_T = e^t - e^{-t}$

$\vec{v}$ and $\vec{a}$ are correct by differentiation.

$$|\vec{v}| = \sqrt{e^{2t} + 2 + e^{-2t}} = e^t + e^{-t} \qquad a_T = \frac{d|\vec{v}|}{dt} = e^t - e^{-t}$$

$$L = \int ds = \int |\vec{v}|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = (e^1 - e^{-1}) - (1 - 1) = e - \frac{1}{e}$$

7. A wire in the shape of the curve $\vec{r}(t) = \left(e^t, \quad \sqrt{2}\,t, \quad e^{-t}\right)$ has linear mass density $\rho = x + z$. Find its total mass between $t = 0$ and $t = 1$.

a. $\frac{e^2}{2} + 1 - \frac{1}{2e^2}$

b. $\frac{e^2}{2} + 2 - \frac{1}{2e^2}$ Correct Choice

c. $\frac{e^2}{2} + 2 + \frac{1}{2e^2}$

d. $e - \frac{1}{e}$

e. $e + \frac{1}{e}$

$$M = \int \rho\,ds = \int (x + z)|\vec{v}|\,dt = \int_0^1 (e^t + e^{-t})(e^t + e^{-t})\,dt = \int_0^1 (e^{2t} + 2 + e^{-2t})\,dt$$

$$= \left[\frac{e^{2t}}{2} + 2t + \frac{e^{-2t}}{-2}\right]_0^1 = \left(\frac{e^2}{2} + 2 + \frac{e^{-2}}{-2}\right) - \left(\frac{1}{2} + \frac{1}{-2}\right) = \frac{e^2}{2} + 2 - \frac{1}{2e^2}$$

8. Find the work done to move an object along the curve $\vec{r}(t) = \left(e^t, \quad \sqrt{2}\,t, \quad e^{-t}\right)$ between $t = 0$ and $t = 1$ by the force $\vec{F} = \langle z, 0, -x\rangle$?

a. $2e - \frac{2}{e}$

b. $2e + \frac{2}{e}$

c. $e - \frac{1}{e}$

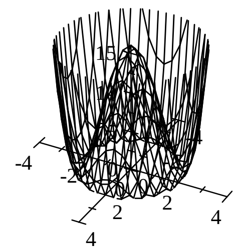
d. $e + \frac{1}{e}$

e. 2 Correct Choice

$$\vec{F}(\vec{r}(t)) = \langle e^{-t}, 0, -e^t\rangle \qquad \vec{v} = \left\langle e^t, \quad \sqrt{2}, \quad -e^{-t}\right\rangle$$

$$W = \int \vec{F} \cdot d\vec{s} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{v}\,dt = \int_0^1 (e^{-t}e^t + e^t e^{-t})\,dt = \int_0^1 2\,dt = \left[2t\right]_0^1 = 2$$

9. The plot at the right is the graph of which function?



a. $f(x,y) = (x^2 + y^2 - 4)^2$ Correct Choice

b. $f(x,y) = (x^2 + y^2)^2 - 16$

c. $f(x,y) = x^2 + y^2 - 4$

d. $f(x,y) = (x - 2)^2 + (y - 2)^2$

e. $f(x,y) = 2x^2 + 2y^2$

The graph is 0 on the circle $x^2 + y^2 = 4$. So the function is not (d) or (e).

The graph is always positive. So the function is not (b) or (c) which are negative when $x = y = 0$.

10. If $z = x^{3e}e^{3y}$ which of the following is FALSE?

a. $\frac{\partial z}{\partial x} = 3ex^{3e-1}e^{3y}$

b. $\frac{\partial z}{\partial y} = 3x^{3e}e^{3y}$

c. $\frac{\partial^2 z}{\partial x^2} = (9e^2 - 3e)x^{3e-2}e^{3y}$

d. $\frac{\partial^2 z}{\partial y \partial x} = 9e^2x^{3e-1}e^{3y}$ Correct Choice

e. $\frac{\partial^2 z}{\partial x \partial y} = 9ex^{3e-1}e^{3y}$

$\frac{\partial^2 z}{\partial y \partial x}$ must equal $\frac{\partial^2 z}{\partial x \partial y}$ and (d) is wrong.

11. Find the plane tangent to the graph of $z = x\ln(y)$ at the point $(2,e)$. Its z-intercept is

a. $-e$

b. -2 Correct Choice

c. 0

d. 2

e. e

$f = x\ln(y)$ $\quad f(2,e) = 2$ $\quad z = f(2,e) + f_x(2,e)(x-2) + f_y(2,e)(y-e)$

$f_x = \ln(y)$ $\quad f_x(2,e) = 1$ $\quad = 2 + 1(x-2) + \frac{2}{e}(y-e)$

$f_y = \frac{x}{y}$ $\quad f_y(2,e) = \frac{2}{e}$ $\quad$ When $x = y = 0$, we have $z = 2 + (-2) + \frac{2}{e}(-e) = -2$.

Work Out: (12 points each. Part credit possible. Show all work.)

12. Find the vector projection of the vector $\vec{a} = \langle 1, 2, 3 \rangle$ along the vector $\vec{b} = \langle 2, 1, -2 \rangle$.

$\vec{a} \cdot \vec{b} = 2 + 2 - 6 = -2$ $\qquad$ $\vec{b} \cdot \vec{b} = 4 + 1 + 4 = 9$

$\text{proj}_{\vec{b}}\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \dfrac{-2}{9} \langle 2, 1, -2 \rangle = \left\langle \dfrac{-4}{9}, \dfrac{-2}{9}, \dfrac{4}{9} \right\rangle$

13. Find the point where the line $\dfrac{x-4}{-1} = \dfrac{y-5}{2} = \dfrac{z-7}{2}$ intersects the plane $x - 3y + z = 6$.

METHOD 1:

The parametric version of the line is $x = 4 - t \quad y = 5 + 2t \quad z = 7 + 2t$.

Substitute into the plane and solve for t:

$6 = x - 3y + z = (4 - t) - 3(5 + 2t) + (7 + 2t) = -4 - 5t \qquad 5t = -10 \qquad t = -2$

Substitute back into the line:

$x = 4 - t = 6 \qquad y = 5 + 2t = 1 \qquad z = 7 + 2t = 3$

The point is $(6, 1, 3)$

METHOD 2:

Multiply the line by –2: $2x - 8 = -y + 5 = -z + 7$

Express y and z in terms of x: $y = -2x + 13 \qquad z = -2x + 15$

Substitute into the plane and solve for x:

$6 = x - 3y + z = x - 3(-2x + 13) + (-2x + 15) = 5x - 24 \qquad 5x = 30 \qquad x = 6$

Substitute back into y and z:

$y = -2x + 13 = 1 \qquad z = -2x + 15 = 3$

The point is $(6, 1, 3)$

14. The pressure, P, volume, V, and temperature, T, of an ideal gas are related by

$$P = \frac{kT}{V} \quad \text{for some constant } k.$$

At a certain instant, for a certain sample $k = 5\frac{\text{cm}^3\text{-atm}}{°\text{K}}$, $V = 1000\,\text{cm}^3$, and $T = 300\,°\text{K}$.

At that instant, the volume and temperature are increasing at $\frac{dV}{dt} = 10\frac{\text{cm}^3}{\text{sec}}$, and $\frac{dT}{dt} = 2\frac{°\text{K}}{\text{sec}}$.

At that instant, what is the pressure, is it increasing or decreasing and at what rate?

$$P = \frac{kT}{V} = \frac{5 \bullet 300}{1000}\frac{\text{cm}^3\text{-atm}}{°\text{K}}\frac{°\text{K}}{\text{cm}^3} = 1.5\,\text{atm}$$

$$\frac{dP}{dt} = \frac{\partial P}{\partial V}\frac{dV}{dt} + \frac{\partial P}{\partial T}\frac{dT}{dt} = \frac{-kT}{V^2}\frac{dV}{dt} + \frac{k}{V}\frac{dT}{dt} = \frac{-5 \bullet 300}{1000^2} \bullet 10 + \frac{5}{1000}2 = -\frac{1}{200} = -0.005\frac{\text{atm}}{\text{sec}}$$

Since $\frac{dP}{dt}$ is negative, the pressure is decreasing.

15. For an adjustable lens, the distance from the lens to the image, v, is related to the distance from the lens to the object, u, and the focal length, f, by the formula

$$\frac{1}{v} = \frac{1}{f} - \frac{1}{u} \qquad \text{or} \qquad v = \frac{fu}{u-f}$$

Currently $f = 4\,\text{cm}$ $u = 6\,\text{cm}$ and so $v = 12\,\text{cm}$

If the focal length is increased by $\Delta f = 0.2\,\text{cm}$, and the distance from the lens to the object is increased by $\Delta u = 0.3\,\text{cm}$, use differentials to estimate how much the image moves.

Does the distance from the lens to the image increase or decrease?

$$\Delta v = \frac{\partial v}{\partial f}\Delta f + \frac{\partial v}{\partial u}\Delta u = \frac{(u-f)u - fu(-1)}{(u-f)^2}\Delta f + \frac{(u-f)f - fu(1)}{(u-f)^2}\Delta u = \frac{u^2}{(u-f)^2}\Delta f + \frac{-f^2}{(u-f)^2}\Delta u$$

$$= \frac{36}{(2)^2}0.2 + \frac{-16}{(2)^2}0.3 = 9 \bullet 0.2 - 4 \bullet 0.3 = 1.8 - 1.2 = 0.6 \qquad \text{increases}$$