Name______________________________ Sec__________

1-14	/56	17	/12
15	/12	18	/12
16	/12		
Total	/104		

MATH 253 Exam 1 Fall 2008

Sections 501-503,200 Solutions P. Yasskin

Multiple Choice: (4 points each. No part credit.)

1. For the triangle with vertices $A = \left(\sqrt{2}, 2, -2\right)$, $B = \left(\sqrt{2}, -1, 1\right)$ and $C = \left(3\sqrt{2}, -3, 3\right)$ find the angle at B.

a. $45°$

b. $60°$

c. $120°$

d. $135°$ correct choice

e. $150°$

$$\overrightarrow{BA} = A - B = (0, 3, -3) \qquad \overrightarrow{BC} = C - B = \left(2\sqrt{2}, -2, 2\right)$$

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = -6 - 6 = -12 \qquad \left|\overrightarrow{BA}\right| = \sqrt{9+9} = 3\sqrt{2} \qquad \left|\overrightarrow{BC}\right| = \sqrt{8+4+4} = 4$$

$$\cos\theta = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right|} = \frac{-12}{3\sqrt{2}\,4} = \frac{-1}{\sqrt{2}} \qquad \theta = 135°$$

2. For the triangle with vertices $A = \left(\sqrt{2}, 2, -2\right)$, $B = \left(\sqrt{2}, -1, 1\right)$ and $C = \left(3\sqrt{2}, -3, 3\right)$ find the area.

a. 144

b. 72

c. 12

d. $\sqrt{72}$

e. 6 correct choice

$$\overrightarrow{BA} = (0, 3, -3) \qquad \overrightarrow{BC} = \left(2\sqrt{2}, -2, 2\right)$$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 3 & -3 \\ 2\sqrt{2} & -2 & 2 \end{vmatrix} = \hat{\imath}(6-6) - \hat{\jmath}\left(0 + 6\sqrt{2}\right) + \hat{k}\left(0 - 6\sqrt{2}\right) = \left(0, -6\sqrt{2}\right.$$

$$Area = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\sqrt{72+72} = \frac{1}{2}\sqrt{144} = 6$$

Problems 3 through 8 refer to the curve $\vec{r}(t) = (t^2, 2t, \ln t)$:

3. Find the line tangent to the curve at the point $(1,2,0)$.

a. $(x,y,z) = (2+t, 2+2t, 1)$

b. $(x,y,z) = (2+t, 2-2t, 1)$

c. $(x,y,z) = (2+t, -2-2t, 1)$

d. $(x,y,z) = (1+2t, 2+2t, t)$ correct choice

e. $(x,y,z) = (1+2t, 2-2t, t)$

$\vec{r}(t) = (t^2, 2t, \ln t) \qquad \vec{v}(t) = \left(2t, 2, \frac{1}{t}\right) \qquad \vec{r}(1) = (1,2,0) \qquad \vec{v}(1) = (2,2,1)$

$X = P + t\vec{v} \qquad (x,y,z) = \vec{r}(1) + t\vec{v}(1) = (1,2,0) + t(2,2,1) = (1+2t, 2+2t, t)$

4. Find the arc length of the curve between $(1,2,0)$ and $(4,4,\ln 2)$.
HINT: Look for a perfect square.

a. $3 + \ln 2$ correct choice

b. $4 + \ln 2$

c. $3 + \ln 4$

d. $4 + \ln 4$

e. $1 + \ln 4$

$\vec{v}(t) = \left(2t, 2, \frac{1}{t}\right) \qquad |\vec{v}| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4 + 4t^2 + 1}{t^2}} = \sqrt{\frac{(2t^2+1)^2}{t^2}} = \frac{2t^2+1}{t} = 2t + \frac{1}{t}$

$(1,2,0) = \vec{r}(1) \qquad (4,4,\ln 2) = \vec{r}(2)$

$L = \int_1^2 |\vec{v}|\,dt = \int_1^2 \left(2t + \frac{1}{t}\right) dt = [t^2 + \ln|t|]_1^2 = (4 + \ln 2) - (1) = 3 + \ln 2$

5. Find the tangential acceleration a_T of the curve.

a. $\frac{2t^2+1}{t^2}$

b. $\frac{2t^2-1}{t^2}$ correct choice

c. $\frac{4t^4+1}{t^2}$

d. $\frac{4t^4-1}{t^2}$

e. $2 + \ln t$

$a_T = \vec{a} \cdot \hat{T} = \frac{1}{|\vec{v}|}\vec{a} \cdot \vec{v} = \frac{t}{2t^2+1}\left(2, 0, -\frac{1}{t^2}\right) \cdot \left(2t, 2, \frac{1}{t}\right) = \frac{t}{2t^2+1}\left(4t - \frac{1}{t^3}\right) = \frac{t(4t^4-1)}{t^3(2t^2+1)} = \frac{2t^2-1}{t^2}$

$a_T = \frac{d|\vec{v}|}{dt} = \frac{d}{dt}\left(2t + \frac{1}{t}\right) = 2 - \frac{1}{t^2} = \frac{2t^2-1}{t^2}$

6. Find the binormal vector $\hat{B}$ of the curve.

a. $\left(\dfrac{-1}{2t^2+1}, \dfrac{-2t}{2t^2+1}, \dfrac{-2t^2}{2t^2+1}\right)$

b. $\left(\dfrac{-1}{2t^2+1}, \dfrac{2t}{2t^2+1}, \dfrac{-2t^2}{2t^2+1}\right)$ correct choice

c. $\left(\dfrac{-2}{t^2}, \dfrac{-4}{t}, -4\right)$

d. $\left(\dfrac{-2}{t^2}, \dfrac{4}{t}, -4\right)$

e. $(-2, -4t, -4t^2)$

$$\vec{v}(t) = \left(2t, 2, \frac{1}{t}\right) \qquad \vec{a}(t) = \left(2, 0, -\frac{1}{t^2}\right)$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2t & 2 & \frac{1}{t} \\ 2 & 0 & -\frac{1}{t^2} \end{vmatrix} = \hat{\imath}\left(-\frac{2}{t^2}\right) - \hat{\jmath}\left(-\frac{2}{t} - \frac{2}{t}\right) + \hat{k}(-4) = \left(-\frac{2}{t^2}, \frac{4}{t}, -4\right)$$

$$|\vec{v}\times\vec{a}| = \sqrt{\frac{4}{t^4} + \frac{16}{t^2} + 16} = \sqrt{4\frac{1+4t^2+4t^4}{t^4}} = 2\frac{2t^2+1}{t^2}$$

$$\hat{B} = \frac{\vec{v}\times\vec{a}}{|\vec{v}\times\vec{a}|} = \frac{t^2}{2(2t^2+1)}\left(-\frac{2}{t^2}, \frac{4}{t}, -4\right) = \left(\frac{-1}{2t^2+1}, \frac{2t}{2t^2+1}, \frac{-2t^2}{2t^2+1}\right)$$

7. Find the mass of a wire in the shape of the curve between $(1,2,0)$ and $(4,4,\ln 2)$, if its linear mass density is $\rho = x + \dfrac{y}{2}e^z$.

a. 8

b. 12

c. 16

d. 18 correct choice

e. 20

$$\rho = x + \frac{y}{2}e^z = t^2 + t\,e^{\ln t} = 2t^2 \qquad |\vec{v}| = 2t + \frac{1}{t}$$

$$M = \int_1^2 \rho|\vec{v}|\,dt = \int_1^2 2t^2\left(2t + \frac{1}{t}\right)dt = \int_1^2 (4t^3 + 2t)\,dt = [t^4 + t^2]_1^2 = (16+4) - (1+1) = 18$$

8. Find the work done to move an object along the curve from $(1,2,0)$ to $(4,4,\ln 2)$, under the action of the force $\vec{F} = (y, x, xy)$.

a. $\dfrac{28}{3}$

b. $\dfrac{32}{3}$

c. $\dfrac{56}{3}$ correct choice

d. $\dfrac{64}{3}$

e. 27

$$\vec{F} = (y, x, xy) = (2t, t^2, 2t^3) \qquad \vec{v}(t) = \left(2t, 2, \frac{1}{t}\right) \qquad \vec{F}\cdot\vec{v} = 4t^2 + 2t^2 + 2t^2 = 8t^2$$

$$W = \int_1^2 \vec{F}\cdot\vec{v}\,dt = \int_1^2 8t^2\,dt = \left[\frac{8}{3}t^3\right]_1^2 = \frac{8}{3}(8-1) = \frac{56}{3}$$

9. Find the plane which passes through the point $P = (2,4,-1)$ and is perpendicular to the line $(x,y,z) = (1+3t, 2+t, 3+2t)$. Its z-intercept is

a. -8

b. -4

c. -2

d. 4 correct choice

e. 8

The normal to the plane is the tangent to the line: $\vec{N} = \vec{v} = (3,1,2)$
$\vec{N} \cdot X = \vec{N} \cdot P$ $\quad 3x + y + 2z = 3(2) + (4) + 2(-1) = 8 \quad z = -\frac{3}{2}x - \frac{1}{2}y + 4$

10. Find the point where the line $(x,y,z) = (1-3t, 2+t, 1-2t)$ intersects the plane $2x + 3y - 3z = -1$. At this point $x + y + z =$

a. 12 correct choice

b. 6

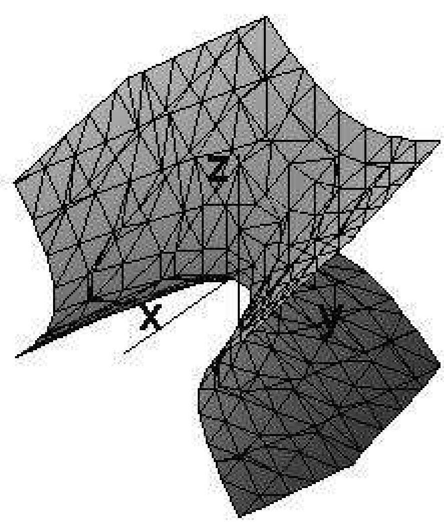
c. 5

d. 4

e. -1

Plug the line into the plane: $2(1-3t) + 3(2+t) - 3(1-2t) = -1 \quad 5 + 3t = -1 \quad t = -2$
Plug $t = -2$ into the line: $(x,y,z) = (1-3(-2), 2+(-2), 1-2(-2)) = (7,0,5)$ So $x + y + z = 12$

11. Which of the following is the equation of the surface?



a. $x^2 - y^2 - z^2 = 0$

b. $x - y^2 + z^2 = 0$ correct choice

c. $x + y^2 - z^2 = 0$

d. $x - y^2 - z^2 = 0$

e. $x + y^2 + z^2 = 0$

This is a hyperbolic paraboloid.
(a) is a cone. (d) and (e) are elliptic paraboloids.
This hyperbolic paraboloid opens up in the xy-plane and down in the xz-plane, as in (b).

12. Which of the following is the function graphed?

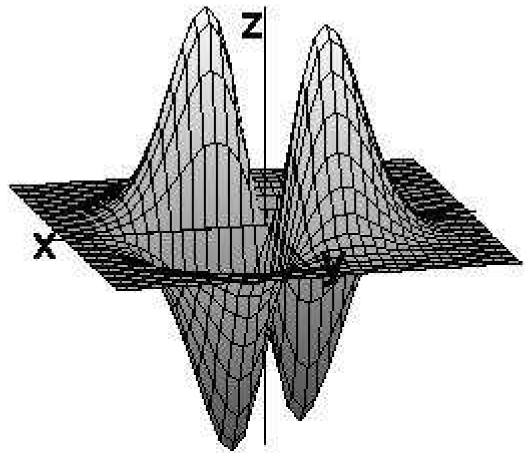
a. $z = x^2y^2e^{-x^2-y^2}$

b. $z = (x^2 - y^2)e^{-x^2-y^2}$

c. $z = (y^2 - x^2)e^{-x^2-y^2}$

d. $z = xye^{-x^2-y^2}$

e. $z = -xye^{-x^2-y^2}$ correct choice



The graph is zero on the axes, negative in quadrants I and III, and positive in quadrants II and IV. (a) is everywhere positive. (b) and (c) are non-zero on the axes. (d) is positive in quadrant I.

13. If $f(x,y) = y\sin(xy)$, which of the following is FALSE?

a. $f_x(1,2) = 4\cos(2)$

b. $f_y(1,2) = \sin(2) + 2\cos(2)$

c. $f_{xx}(1,2) = -4\sin(2) + 4\cos(2)$ correct choice

d. $f_{xy}(1,2) = 4\cos(2) - 4\sin(2)$

e. $f_{yy}(1,2) = 2\cos(2) - 2\sin(2)$

$f_x(x,y) = y^2\cos(xy)$ $f_y(x,y) = \sin(xy) + xy\cos(xy)$
$f_{xx}(x,y) = -y^3\sin(xy)$ $f_{xy}(x,y) = 2y\cos(xy) - xy^2\sin(xy)$ $f_{yy}(x,y) = 2x\cos(xy) - x^2y\sin(xy)$
$f_x(1,2) = 4\cos(2)$ $f_y(1,2) = \sin(2) + 2\cos(2)$
$f_{xx}(1,2) = -8\sin(2)$ $f_{xy}(1,2) = 4\cos(2) - 4\sin(2)$ $f_{yy}(1,2) = 2\cos(2) - 2\sin(2)$

14. A function $f(x,y)$ satisfies: $f(3,4) = 2$, $f_x(3,4) = -2$, $f_y(3,4) = 3$.
Use the linear approximation to estimate $f(3.2,3.9)$.

a. -0.7

b. 1.3 correct choice

c. 2.7

d. 5.3

e. 7.3

$f(x,y) \approx f(3,4) + f_x(3,4)(x-3) + f_y(3,4)(y-4) = 2 - 2(x-3) + 3(y-4)$
$f(3.2,3.9) \approx 2 - 2(3.2-3) + 3(3.9-4) = 2 - .4 - .3 = 1.3$

Work Out: (Points indicated. Part credit possible. Show all work.)

15. (12 points) Duke Skywater is flying across the galaxy in the Millenium Eagle when he find himself passing through a dangerous polaron field. He is currently at the point $\vec{r} = (-1,1,2)$ and has velocity $\vec{v} = (0.1,-0.2,0.2)$ and the polaron density is $\rho = xz^2 + yz^3$.

a. (8 pts) What is the polaron density and its rate of change as currently seen by Duke?

$\rho(-1,1,2) = -4+8 = 4$

$\vec{\nabla}\rho = (z^2, z^3, 2xz+3yz^2) \qquad \vec{\nabla}\rho(-1,1,2) = (4,8,8)$

$\dfrac{d\rho}{dt} = \vec{v}\cdot\vec{\nabla}\rho = (0.1,-0.2,0.2)\cdot(4,8,8) = 0.4-1.6+1.6 = 0.4$

b. (4 pts) In what unit vector direction should Duke travel to **reduce** the polaron density as fast as possible?

The polaron decreases fastest in the direction opposite to the gradient:

$\vec{w} = -\vec{\nabla}\rho(-1,1,2) = (-4,-8,-8) \qquad |\vec{w}| = \sqrt{16+64+64} = \sqrt{144} = 12$

The unit vector direction is

$\hat{w} = \dfrac{\vec{w}}{|\vec{w}|} = \dfrac{1}{12}(-4,-8,-8) = \left(-\dfrac{1}{3},-\dfrac{2}{3},-\dfrac{2}{3}\right)$

16. (12 points) The temperature around a candle is given by $T = 110 - x^2 - y^2 - 2z^2$.

Find the maximum temperature on the plane $4x+6y+8z = 42$ and the point where it occur.

METHOD 1: Lagrange Multipliers:

$\vec{\nabla}T = (-2x,-2y,-4z) \qquad g = 4x+6y+8z \qquad \vec{\nabla}g = (4,6,8)$

Lagrange equations: $\vec{\nabla}T = \lambda\vec{\nabla}g \qquad (-2x,-2y,-4z) = \lambda(4,6,8)$

$-2x = 4\lambda \qquad -2y = 6\lambda \qquad -4z = 8\lambda$

$x = -2\lambda \qquad y = -3\lambda \qquad z = -2\lambda$ Plug into constraint, $g = 42$:

$4(-2\lambda)+6(-3\lambda)+8(-2\lambda) = 42 \qquad -42\lambda = 42 \qquad \lambda = -1$

$(x,y,z) = (2,3,2) \qquad T = 110-4-9-8 = 89$

METHOD 2: Eliminate a variable:

$4x+6y+8z = 42 \qquad x = \dfrac{21}{2} - \dfrac{3}{2}y - 2z \qquad T = 110 - \left(\dfrac{21}{2} - \dfrac{3}{2}y - 2z\right)^2 - y^2 - 2z^2$

$T_y = -2\left(\dfrac{21}{2} - \dfrac{3}{2}y - 2z\right)\left(-\dfrac{3}{2}\right) - 2y = 0 \qquad T_z = -2\left(\dfrac{21}{2} - \dfrac{3}{2}y - 2z\right)(-2) - 4z = 0$

$\dfrac{63}{2} - \dfrac{13}{2}y - 6z = 0 \qquad 42 - 6y - 12z = 0$

$13y+12z = 63 \qquad 6y+12z = 42$ subtract $7y = 21 \qquad y = 3$

$12z = 42-6y = 42-18 = 24 \qquad z = 2$

$x = \dfrac{21}{2} - \dfrac{3}{2}y - 2z = \dfrac{21}{2} - \dfrac{9}{2} - 4 = 2$

$(x,y,z) = (2,3,2) \qquad T = 110-4-9-8 = 89$

17. (12 points) Find an equation of the plane tangent to the graph of the function $f(x,y) = 2x^2y - xy^2$ at $(x,y) = (2,1)$. Then find its z-intercept.

$f(x,y) = 2x^2y - xy^2 \qquad f_x(x,y) = 4xy - y^2 \qquad f_y(x,y) = 2x^2 - 2xy$

$f(2,1) = 8 - 2 = 6 \qquad f_x(2,1) = 8 - 1 = 7 \qquad f_y(2,1) = 8 - 4 = 4$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 6 + 7(x-2) + 4(y-1) = 7x + 4y - 12$

The z-intercept is $c = -12$.

18. (12 points) Find an equation of the plane tangent to the surface $x^4 + x^2y^2 + y^2z^2 = 21$ at the point $(x,y,z) = (1,-2,2)$. Then find its z-intercept.

$F = x^4 + x^2y^2 + y^2z^2 \qquad \vec{\nabla}F = (4x^3 + 2xy^2, 2x^2y + 2yz^2, 2y^2z) \qquad \vec{N} = \vec{\nabla}F(1,-2,2) = (12,-20,16)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad 12x - 20y + 16z = 12(1) - 20(-2) + 16(2) = 84 \qquad 3x - 5y + 4z = 21$

The z-intercept is $c = \frac{21}{4}$.