Name________________________ Sec__________

MATH 253 Final Exam Fall 2008

Sections 501-503,200 Solutions P. Yasskin

1-10	/50	12	/25
11	/15	13	/15
Total			/105

Multiple Choice: (5 points each. No part credit.)

1. Find a parametric equation of the line tangent to the curve $\vec{r}(\theta) = (2\sin\theta, 2\cos\theta, \theta)$ at the point $(0,-2,\pi)$.

a. $X(t) = (-2,-2t,1+\pi t)$

b. $X(t) = (0,-2t-2,\pi+t)$

c. $X(t) = (-2t,-2,\pi+t)$ correct choice

d. $X(t) = (-2t-2,0,\pi+t)$

e. $X(t) = (0,-2t-2,1+\pi t)$

$\vec{r}(\theta) = (2\sin\theta, 2\cos\theta, \theta) = (0,-2,\pi)$ at $\theta = \pi$.
$\vec{v}(\theta) = (2\cos\theta, -2\sin\theta, 1)$ $\vec{v}(\pi) = (-2,0,1)$
$X(t) = (0,-2,\pi) + t(-2,0,1) = (-2t,-2,\pi+t)$

2. The density of the fog is given by $\rho = 30 - x^2 - y^2 - z$. If an airplane is at the position $(x,y,z) = \left(\sqrt{2},2,4\right)$, in what unit vector direction should the airplane initially travel to get out of the fog as quickly as possible?

a. $\left(\frac{-2\sqrt{2}}{5}, \frac{-4}{5}, \frac{-1}{5}\right)$

b. $\left(\frac{2\sqrt{2}}{5}, \frac{4}{5}, \frac{1}{5}\right)$ correct choice

c. $\left(\frac{-\sqrt{2}}{\sqrt{10}}, \frac{-2}{\sqrt{10}}, \frac{-2}{\sqrt{10}}\right)$

d. $\left(\frac{\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

e. $\left(\frac{-\sqrt{2}}{\sqrt{10}}, \frac{2}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right)$

$\vec{\nabla}\rho = (-2x,-2y,-1)$ $\vec{\nabla}\rho\left(\sqrt{2},2,4\right) = \left(-2\sqrt{2},-4,-1\right)$
To decrease the density, the airplane should travel in the direction $-\vec{\nabla}\rho\left(\sqrt{2},2,4\right) = \left(2\sqrt{2},4,1\right)$.
Since $\left|-\vec{\nabla}\rho\right| = \sqrt{8+16+1} = 5$, the unit vector direction is $\left(\frac{2\sqrt{2}}{5}, \frac{4}{5}, \frac{1}{5}\right)$.

3. Find an equation of the plane tangent to the graph of the function $z = x^2y + xy^2$ at the point $(2,1)$.

a. $z = 5x + 8y + 6$

b. $z = -5x - 8y + 6$

c. $z = -5x - 8y + 24$

d. $z = 5x + 8y - 12$ correct choice

e. $z = 5x + 8y - 6$

$f(x,y) = x^2y + xy^2 \qquad f_x(x,y) = 2xy + y^2 \qquad f_y(x,y) = x^2 + 2xy$

$f(2,1) = 6 \qquad f_x(2,1) = 5 \qquad f_y(2,1) = 8$

$z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) = 6 + 5(x-2) + 8(y-1) = 5x + 8y - 12$

4. Find an equation of the plane tangent to the level surface $x^2y^2 + x^2z^2 + y^2z^2 = 49$ at the point $(1,2,3)$.

a. $13x + 20y + 15z = 98$ correct choice

b. $13x + 10y + 5z = 48$

c. $13x - 10y + 5z = 8$

d. $13x - 20y + 15z = 18$

e. $39x + 20y + 5z = 94$

$f = x^2y^2 + x^2z^2 + y^2z^2 \qquad P = (1,2,3)$

$\vec{\nabla} f = (2xy^2 + 2xz^2, 2yx^2 + 2yz^2, 2zx^2 + 2zy^2) \qquad \vec{N} = \vec{\nabla} f(P) = (26,40,30)$

$\vec{N} \cdot X = \vec{N} \cdot P \qquad 26x + 40y + 30z = 26 \cdot 1 + 40 \cdot 2 + 30 \cdot 3 = 196 \qquad 13x + 20y + 15z = 98$

5. Find the equation of the plane tangent to the parametric surface $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, \theta)$ at the point where $(r,\theta) = (2,\pi)$.

a. $-x = -2y + z$

b. $-x + 2y - z = -2 + \pi$

c. $-x - 2y - z = -2 + \pi$

d. $-y + 2z = 2\pi$

e. $y + 2z = 2\pi$ correct choice

$P = \vec{R}(2,\pi) = (2\cos\pi, 2\sin\pi, \pi) = (-2,0,\pi)$

$\vec{e}_r = (\cos\theta, \sin\theta, 0) \qquad \vec{e}_r(2,\pi) = (\cos\pi, \sin\pi, 0) = (-1,0,0)$

$\vec{e}_\theta = (-r\sin\theta, r\cos\theta, 1) \qquad \vec{e}_\theta(2,\pi) = (-2\sin\pi, 2\cos\pi, 1) = (0,-2,1)$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \hat{\imath}(0) - \hat{\jmath}(-1) + \hat{k}(2) = (0,1,2) \qquad \vec{N} \cdot X = \vec{N} \cdot P \qquad y + 2z = 2\pi$$

6. A satellite is travelling from East to West directly above the equator. In what direction does the binormal $\hat{B}$ point?

- **a**. North
- **b**. South correct choice
- **c**. Up
- **d**. Down
- **e**. West

$\hat{T}$ points West, $\hat{N}$ points Down. So $\hat{B} = \hat{T} \times \hat{N}$ points South by the right hand rule.

7. Compute $\int_P^Q 2x\,dx + 2y\,dy + 2z\,dz$ along the straight line from $P = (1,-2,2)$ to $Q = (3,-4,12)$.
HINT: Use the Fundamental Theorem of Calculus for Curves.

- **a**. -10
- **b**. $\sqrt{10}$
- **c**. 10
- **d**. 108
- **e**. 160 correct choice

Let $\vec{F} = (2x,2y,2z)$. Then $\vec{F} = \vec{\nabla} f$ where $f = x^2 + y^2 + z^2$. So by the FTCC,

$$\int_P^Q 2x\,dx + 2y\,dy + 2z\,dz = \int_P^Q \vec{F}\,d\vec{s} = \int_P^Q \vec{\nabla} f\,d\vec{s} = f(Q) - f(P) = (9 + 16 + 144) - (1 + 4 + 4) = 160$$

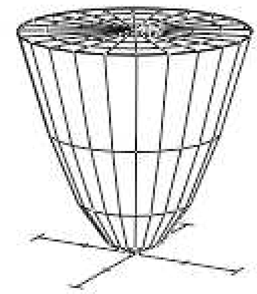
8. Compute $\oint 2x\,dx + 2xy\,dy$ counterclockwise around the boundary of the rectangle $2 \le x \le 4$, $1 \le y \le 4$.
HINT: Use Green's Theorem.

- **a**. 6
- **b**. 18
- **c**. 30 correct choice
- **d**. 36
- **e**. 72

By Green's Theorem:

$$\oint 2x\,dx + 2xy\,dy = \int_1^4\int_2^4 \partial_x(2xy) - \partial_y(2x)\,dx\,dy = \int_1^4\int_2^4 2y\,dx\,dy = \int_2^4 1\,dx\int_1^4 2y\,dy = 2\Big[y^2\Big]_1^4 = 30$$

9. Compute $\iint\limits_{\partial V} \vec{F} \cdot d\vec{S}$ over the complete boundary of the solid above the paraboloid $z = x^2 + y^2$ and below the plane $z = 4$ with outward normal, for the vector field $\vec{F} = (xy^2, yx^2, z^2)$.
HINT: Use Gauss' Theorem.



a. $-\frac{128}{3}\pi$

b. 40π

c. 72π

d. $\frac{160}{3}\pi$ correct choice

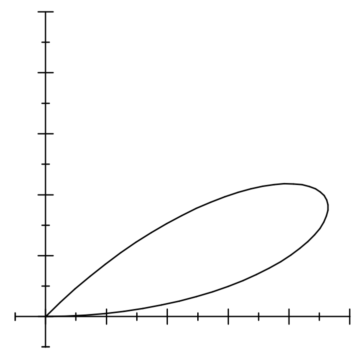
e. $\frac{896}{15}\pi$

$\vec{\nabla} \cdot \vec{F} = y^2 + x^2 + 2z = r^2 + 2z$ in cylindrical coordinates.

$$\iint\limits_{\partial V} \vec{F} \cdot d\vec{S} = \iiint\limits_{V} \vec{\nabla} \cdot \vec{F}\, dV = \int_0^{2\pi}\int_0^2\int_{r^2}^4 (r^2 + 2z) r\, dz\, dr\, d\theta = 2\pi \int_0^2 \Big[r^3 z + z^2 r \Big]_{z=r^2}^4 dr = 2\pi \int_0^2 (4r^3 + 16r) - (2r^5)\, dr$$

$$= 2\pi \left[r^4 + 8r^2 - \frac{r^6}{3} \right]_0^2 = 2\pi\left(16 + 32 - \frac{64}{3}\right) = 32\pi\left(3 - \frac{4}{3}\right) = \frac{5 \cdot 32\pi}{3} = \frac{160}{3}\pi$$

10. Find the area of one petal of the 4 leaf rose $r = \sin(4\theta)$.
The petal in the first quadrant.is shown.



a. $\frac{\pi}{16}$ correct choice

b. $\frac{\pi}{8}$

c. $\frac{\pi}{4}$

d. $\frac{\pi}{2}$

e. π

$r = \sin(4\theta) = 0$ at $4\theta = \pi$ or $\theta = \frac{\pi}{4}$

$$A = \iint 1\, dA = \int_0^{\pi/4}\int_0^{\sin(4\theta)} r\, dr\, d\theta = \int_0^{\pi/4} \left[\frac{r^2}{2} \right]_0^{\sin(4\theta)} d\theta = \frac{1}{2}\int_0^{\pi/4} \sin^2(4\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1 - \cos(8\theta)}{2}\, d\theta = \frac{1}{4}\left[\theta - \frac{\sin(8\theta)}{8} \right]_0^{\pi/4} = \frac{\pi}{16}$$

Work Out: (Points indicated. Part credit possible. Show all work.)

11. (15 points) Find the point in the first octant on the graph of $4x^4y^2z = 1$ which is closest to the origin.
HINTS: What is the square of the distance from a point to the origin? Lagrange multipliers are easier.

Minimize $f = x^2 + y^2 + z^2$ subject to $g = 4x^4y^2z = 1$.

Method 1: Lagrange multipliers:

$$\vec{\nabla}f = (2x, 2y, 2z) \qquad \vec{\nabla}g = (16x^3y^2z, 8x^4yz, 4x^4y^2)$$

$$\vec{\nabla}f = \lambda\vec{\nabla}g \quad \Rightarrow \quad 2x = \lambda 16x^3y^2z, \qquad 2y = \lambda 8x^4yz, \qquad 2z = \lambda 4x^4y^2$$

$$\lambda = \frac{2x}{16x^3y^2z} = \frac{2y}{8x^4yz} = \frac{2z}{4x^4y^2} \quad \Rightarrow \quad x^2 = 4z^2, \qquad y^2 = 2z^2 \quad \Rightarrow \quad x = 2z, \qquad y = \sqrt{2}\,z$$

$$1 = 4x^4y^2z = 4(4z^2)^2(2z^2)z = 128z^7 \quad \Rightarrow \quad z = \frac{1}{2} \qquad (x,y,z) = \left(1, \frac{\sqrt{2}}{2}, \frac{1}{2}\right)$$

Method 2: Eliminate a variable:

$$z = \frac{1}{4x^4y^2} \qquad f = x^2 + y^2 + \frac{1}{16x^8y^4}$$

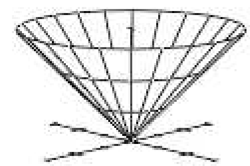
$$f_x = 2x - \frac{1}{2x^9y^4} = 0 \qquad f_y = 2y - \frac{1}{4x^8y^5} = 0 \quad \Rightarrow \quad 4x^{10}y^4 = 1 \qquad 8x^8y^6 = 1$$

equate: $x^2 = 2y^2 \quad \Rightarrow \quad x = \sqrt{2}\,y$ substitute back: $4(2y^2)^5y^4 = 1 \quad \Rightarrow \quad y^{14} = \frac{1}{2^7}$

$$\Rightarrow \quad y = \frac{1}{\sqrt{2}} \qquad (x,y,z) = \left(1, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$

12. (25 points) Verify Stokes' Theorem $\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \oint_{\partial C} \vec{F} \cdot d\vec{s}$

for the cone C given by $z^2 = x^2 + y^2$ for $z \le 2$



oriented down and out, and the vector field $\vec{F} = (yz^2, -xz^2, z^3)$.

Be sure to check and explain the orientations.

Use the following steps:

a. Note: The cone may be parametrized as $\vec{R}(r,\theta) = (r\cos\theta, r\sin\theta, r)$

Compute the surface integral by successively finding:

$$\vec{e}_r,\quad \vec{e}_\theta,\quad \vec{N},\quad \vec{\nabla} \times \vec{F},\quad \vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right),\quad \iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S}$$

$$\begin{array}{r} \\ \vec{e}_r = \\ \vec{e}_\theta = \end{array} \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ (\cos\theta & \sin\theta, & 1\ ) \\ (-r\sin\theta, & r\cos\theta, & 0\ ) \end{array} \right|$$

$$\vec{N} = \vec{e}_r \times \vec{e}_\theta = \hat{\imath}(-r\cos\theta) - \hat{\jmath}(r\sin\theta) + \hat{k}(r\cos^2\theta + r\sin^2\theta) = (-r\cos\theta, -r\sin\theta, r)$$

Reverse $\vec{N} = (r\cos\theta, r\sin\theta, -r)$

$$\vec{\nabla} \times \vec{F} = \left| \begin{array}{ccc} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ yz^2 & -xz^2 & z^3 \end{array} \right| = \hat{\imath}(0 - -2xz) - \hat{\jmath}(0 - 2yz) + \hat{k}(-z^2 - z^2) = (2xz, 2yz, -2z^2)$$

$$\vec{\nabla} \times \vec{F}\left(\vec{R}(r,\theta)\right) = (2r^2\cos\theta, 2r^2\sin\theta, -2r^2)$$

$$\iint_C \vec{\nabla} \times \vec{F} \cdot d\vec{S} = \iint_C \vec{\nabla} \times \vec{F} \cdot \vec{N}\, dr\, d\theta = \int_0^{2\pi}\int_0^2 (2r^3\cos^2\theta + 2r^3\sin^2\theta + 2r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r^3)\, dr\, d\theta = 2\pi[r^4]_0^2 = 32\pi$$

b. Compute the line integral by parametrizing the boundary curve and successively finding:

$$\vec{r}(\theta),\quad \vec{v},\quad \vec{F}(\vec{r}(\theta)),\quad \oint_{\partial C} \vec{F} \cdot d\vec{s}$$

$$\vec{r}(\theta) = (2\cos\theta, 2\sin\theta, 2)$$

$$\vec{v} = (-2\sin\theta, 2\cos\theta, 0)$$

Reverse $\vec{v} = (2\sin\theta, -2\cos\theta, 0)$

$$\vec{F}(\vec{r}(\theta)) = (yz^2, -xz^2, z^3) = (8\sin\theta, -8\cos\theta, 8)$$

$$\oint_{\partial C} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F} \cdot \vec{v}\, d\theta = \int_0^{2\pi} (16\sin^2\theta + 16\cos^2\theta)\, d\theta = \int_0^{2\pi} 16\, d\theta = 32\pi$$

13. (15 points) Find the mass and center of mass of the $\frac{1}{8}$ of the sphere $x^2 + y^2 + z^2 \leq 4$ in the first octant if the density is $\delta = x^2 + y^2 + z^2$.

$$M = \iiint \delta \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \left[-\cos\varphi \right]_0^{\pi/2} \left[\frac{\rho^5}{5} \right]_0^2 = \frac{16}{5}\pi$$

By symmetry, $\bar{x} = \bar{y} = \bar{z}$. So

$$M_{xy} = \iiint z\delta \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho \cos(\varphi) \rho^2 \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta = \frac{\pi}{2} \left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/2} \left[\frac{\rho^6}{6} \right]_0^2 = \frac{8}{3}\pi$$

$$\bar{x} = \bar{y} = \bar{z} = \frac{M_{xy}}{M} = \frac{8\pi}{3} \frac{5}{16\pi} = \frac{5}{6}$$