Name________________________ Sec__________

1-9	/54	11c	/10
10	/10	11d	/10
11a	/10	11e	/ 6
11b	/10	Total	/104

MATH 253 Exam 2 Spring 2009

Sections 200,501,502 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. Find the volume under $z = xy^2$ above the rectangle $1 \le x \le 2$ and $0 \le y \le 2$.

a. 2

b. 4 Correct Choice

c. 6

d. 12

e. $\frac{16}{3}$

$$V = \int_0^2 \int_1^2 xy^2\,dx\,dy = \left[\frac{x^2}{2}\right]_{x=1}^2 \left[\frac{y^3}{3}\right]_{y=0}^2 = \left(\frac{4-1}{2}\right)\left(\frac{8}{3}\right) = 4$$

2. (Non-Honors Only) Compute $\int_0^2 \int_0^y \int_x^y x\,dz\,dx\,dy$.

a. $\frac{1}{2}$

b. $\frac{2}{3}$ Correct Choice

c. $\frac{3}{4}$

d. $\frac{4}{5}$

e. $\frac{5}{6}$

$$\int_0^2 \int_0^y \int_x^y x\,dz\,dx\,dy = \int_0^2 \int_0^y \left[xz\right]_{z=x}^y dx\,dy = \int_0^2 \int_0^y (xy - x^2)\,dx\,dy = \int_0^2 \left[\frac{x^2 y}{2} - \frac{x^3}{3}\right]_{x=0}^y dy$$

$$= \int_0^2 \left(\frac{y^3}{2} - \frac{y^3}{3}\right) dy = \int_0^2 \frac{y^3}{6}\,dy = \left[\frac{y^4}{24}\right]_{y=0}^2 = \frac{16}{24} = \frac{2}{3}$$

3. Compute $\iiint_R z\,dV$ over the region R in the first octant bounded by $y = 9 - x^2$, $z = 2$ and the coordinate planes.

a. 36 Correct Choice

b. 54

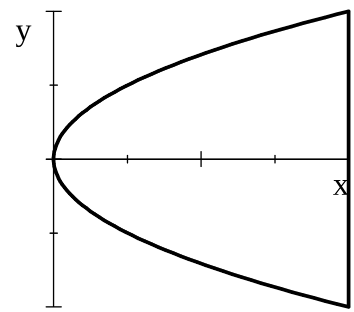
c. 72

d. 96

e. 108

$$\iiint_R z\,dV = \int_0^2\int_0^3\int_0^{9-x^2} z\,dy\,dx\,dz = \int_0^2 z\,dz\int_0^3\int_0^{9-x^2} 1\,dy\,dx = \left[\frac{z^2}{2}\right]_0^2\int_0^3(9-x^2)\,dx = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 36$$

4. Find the mass of the plate bounded by the curves $x = y^2$ and $x = 4$, if the surface mass density is $\rho = x$.



a. $\frac{32}{3}$

b. $\frac{64}{3}$

c. $\frac{128}{3}$

d. $\frac{64}{5}$

e. $\frac{128}{5}$ Correct Choice

$$M = \int_{-2}^2\int_{y^2}^4 x\,dx\,dy = \int_{-2}^2\left[\frac{x^2}{2}\right]_{x=y^2}^4 dy = \int_{-2}^2\left(8 - \frac{y^4}{2}\right)dy = \left[8y - \frac{y^5}{10}\right]_{y=-2}^2 = 2\left(16 - \frac{16}{5}\right) = \frac{128}{5}$$

5. Find the center of mass of the plate bounded by the curves $x = y^2$ and $x = 4$, if the surface mass density is $\rho = x$.

a. $(\bar{x},\bar{y}) = \left(\frac{12}{7},0\right)$

b. $(\bar{x},\bar{y}) = (2,0)$

c. $(\bar{x},\bar{y}) = \left(\frac{20}{7},0\right)$ Correct Choice

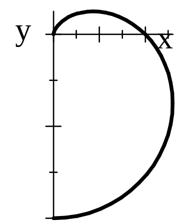
d. $(\bar{x},\bar{y}) = \left(\frac{24}{7},0\right)$

e. $(\bar{x},\bar{y}) = \left(\frac{512}{7},0\right)$

$$M_y = \int_{-2}^2\int_{y^2}^4 x^2\,dx\,dy = \int_{-2}^2\left[\frac{x^3}{3}\right]_{x=y^2}^4 dy = \frac{1}{3}\int_{-2}^2(64 - y^6)\,dy = \frac{1}{3}\left[64y - \frac{y^7}{7}\right]_{y=-2}^2 = \frac{2}{3}\left(128 - \frac{128}{7}\right) = \frac{512}{7}$$

$$\bar{x} = \frac{M_y}{M} = \frac{512}{7}\cdot\frac{5}{128} = \frac{20}{7}$$ $\bar{y} = 0$ by symmetry.

6. A styrofoam board is cut in the shape of the right half of the cardioid $r = 1 - \sin\theta$.
A static electricity charge is put on the board whose surface charge density is given by $\rho_e = x$.
Find the total charge on the board $Q = \iint \rho_e \, dA$.



a. 0

b. $\frac{2}{3}$

c. $\frac{4}{3}$ Correct Choice

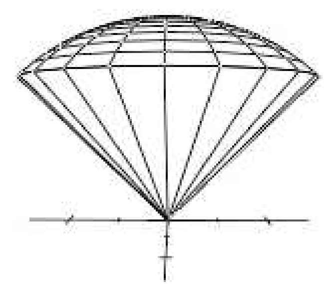
d. $\frac{8\pi}{3}$

e. $\frac{16\pi}{3}$

$$Q = \iint \rho_e \, dA = \int_{-\pi/2}^{\pi/2} \int_0^{1-\sin\theta} r\cos\theta \, r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \cos\theta \right]_0^{1-\sin\theta} d\theta = \int_{-\pi/2}^{\pi/2} \frac{(1-\sin\theta)^3}{3} \cos\theta \, d\theta$$

$$= \left[\frac{-(1-\sin\theta)^4}{12} \right]_{-\pi/2}^{\pi/2} = \frac{-0}{12} - \frac{-2^4}{12} = \frac{4}{3}$$

7. Find the volume of the solid above the cone $z = \sqrt{x^2 + y^2}$ below the hemisphere $x^2 + y^2 + z^2 = 4$.



a. $\frac{8\pi}{3}$

b. $\frac{8\pi}{3}\sqrt{2}$

c. $\frac{16\pi}{3}$

d. $\frac{8\pi}{3}\left(2 - \sqrt{2}\right)$ Correct Choice

e. $\frac{8\pi}{3}\left(2 + \sqrt{2}\right)$

In cylindrical coordinates, the cone is $z = r$ and the hemisphere is $z = \sqrt{4 - r^2}$.
They intersect when $z = r = \sqrt{4 - r^2}$, or $r^2 = 4 - r^2$, or $r = \sqrt{2}$.

$$V = \iiint 1 \, dV = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = 2\pi \int_0^{\sqrt{2}} \left[rz \right]_{z=r}^{\sqrt{4-r^2}} dr = 2\pi \int_0^{\sqrt{2}} \left(r\sqrt{4-r^2} - r^2 \right) dr$$

$$= 2\pi \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^3}{3} \right]_0^{\sqrt{2}} = 2\pi \left(-\frac{2^{3/2}}{3} - \frac{\sqrt{2}^{\,3}}{3} \right) - 2\pi \left(-\frac{(4)^{3/2}}{3} \right) = \frac{2\pi}{3}\left(8 - 4\sqrt{2}\right)$$

In spherical coordinates, the cone is $z = r$ or $\rho\cos\varphi = \rho\sin\varphi$ or $\varphi = \frac{\pi}{4}$
and the hemisphere is $\rho = 2$.

$$V = \iiint 1 \, dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta = 2\pi \left[-\cos\varphi \right]_0^{\pi/4} \left[\frac{\rho^3}{3} \right]_0^2 = \frac{16\pi}{3} \left(-\frac{\sqrt{2}}{2} + 1 \right) = \frac{8\pi}{3}\left(2 - \sqrt{2}\right)$$

8. Compute $\int_0^2 \int_{y^2}^4 y e^{x^2}\, dx\, dy$. HINT: Interchange the order of integration.

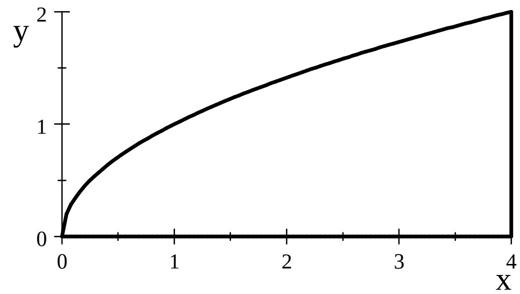
a. $\frac{1}{4}e^{16}$

b. $\frac{1}{4}(e^{16}-1)$ Correct Choice

c. $\frac{1}{2}(1-e^{16})$

d. $\frac{1}{4}(e^{4}-1)$

e. $\frac{1}{2}(1-e^{4})$



$$\int_0^2 \int_{y^2}^4 y e^{x^2}\, dx\, dy = \int_0^4 \int_0^{\sqrt{x}} y e^{x^2}\, dy\, dx = \int_0^4 \left[\frac{y^2}{2} \right]_{y=0}^{\sqrt{x}} e^{x^2}\, dx = \frac{1}{2}\int_0^4 x e^{x^2}\, dx = \left[\frac{1}{4} e^{x^2} \right]_{x=0}^{4} = \frac{1}{4}(e^{16}-1)$$

9. Compute $\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z\cos\left[(x^2+y^2+z^2)^2\right] dz\, dy\, dx$. HINT: Convert to spherical coordinates.

a. $\frac{\pi}{8}\sin(4)$

b. $\frac{\pi}{16}\sin(4)$

c. $\frac{\pi}{4}\sin(16)$

d. $\frac{\pi}{8}\sin(16)$

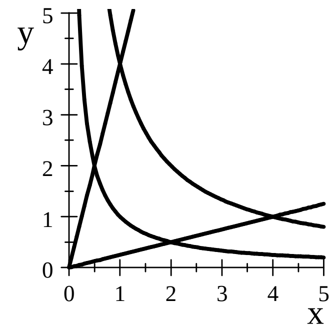
e. $\frac{\pi}{16}\sin(16)$ Correct Choice

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-y^2}} z\cos\left[(x^2+y^2+z^2)^2\right] dz\, dy\, dx = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 \rho\cos(\varphi)\cos(\rho^4)\rho^2\sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \left(\frac{\pi}{2}\right)\left[\frac{\sin^2\varphi}{2} \right]_0^{\pi/2} \int_0^2 \rho^3\cos(\rho^4)\, d\rho = \left(\frac{\pi}{2}\right)\left(\frac{1}{2}\right)\left[\frac{\sin(\rho^4)}{4} \right]_0^2 = \frac{\pi}{16}\sin(16)$$

10. (10 points) Compute $\iint y\,dx\,dy$ over the diamond shaped region bounded by the curves

$$y = 4x \qquad y = \frac{x}{4} \qquad y = \frac{1}{x} \qquad y = \frac{4}{x}$$



HINT: Let $u^2 = xy$ and $v^2 = \frac{y}{x}$. Solve for x and y.

The boundaries are:

$u^2 = xy = 1$ or $u = 1$ $\qquad$ $u^2 = xy = 4$ or $u = 2$

$v^2 = \frac{y}{x} = 4$ or $v = 2$ $\qquad$ $v^2 = \frac{y}{x} = \frac{1}{4}$ or $v = \frac{1}{2}$

$$u^2v^2 = (xy)\left(\frac{y}{x}\right) = y^2 \qquad \frac{u^2}{v^2} = (xy)\left(\frac{x}{y}\right) = x^2 \qquad \text{So} \qquad x = \frac{u}{v} \qquad y = uv$$

$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{1}{v} & v \\ \frac{-u}{v^2} & u \end{vmatrix} = \left|\frac{u}{v} - -\frac{uv}{v^2}\right| = \frac{2u}{v}$$

The integrand is $y = uv$. So

$$\iint y\,dx\,dy = \iint y\,J\,du\,dv = \iint uv\,\frac{2u}{v}\,du\,dv = \int_{1/2}^{2}\int_{1}^{2} 2u^2\,du\,dv = \left[\frac{2u^3}{3}\right]_{u=1}^{2}\left[v\right]_{v=1/2}^{2}$$

$$= \left(\frac{16}{3} - \frac{2}{3}\right)\left(2 - \frac{1}{2}\right) = 7$$

11. Consider the surface, S, given parametrically by

$$\vec{R}(p,q) = \left(\frac{1}{2}p^2, q^2, pq\right) \quad \text{for} \quad 0 \le p \le 3 \quad \text{and} \quad 0 \le q \le 2.$$

a. (10 points) Find $\vec{e}_p$, $\vec{e}_q$, $\vec{N}$, and $\left|\vec{N}\right|$. Simplify $\left|\vec{N}\right|$ by looking for a perfect square.

$$\begin{array}{r|ccc|} & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \vec{e}_p = & (p & 0 & q) \\ \vec{e}_q = & (0 & 2q & p) \end{array}$$

$$\vec{N} = \vec{e}_p \times \vec{e}_q = \hat{\imath}(-2q^2) - \hat{\jmath}(p^2) + \hat{k}(2pq) = (-2q^2, -p^2, 2pq)$$

$$\left|\vec{N}\right| = \sqrt{(-2q^2)^2 + (-p^2)^2 + (2pq)^2} = \sqrt{4q^4 + p^4 + 4p^2q^2} = \sqrt{(p^2 + 2q^2)^2} = p^2 + 2q^2$$

b. (10 points) Compute the surface area of the surface, S.
HINT: $A = \iint 1\, dS$

$$A = \iint \left|\vec{N}\right| dp\,dq = \int_0^2 \int_0^3 (p^2 + 2q^2)\, dp\,dq = \int_0^2 \left[\frac{p^3}{3} + 2pq^2\right]_{p=0}^3 dq = \int_0^2 (9 + 6q^2)\, dq$$

$$= \left[9q + 2q^3\right]_{q=0}^2 = (18 + 16) = 34 \qquad \text{OR}$$

$$A = \iint \left|\vec{N}\right| dq\,dp = \int_0^3 \int_0^2 (p^2 + 2q^2)\, dq\,dp = \int_0^3 \left[p^2 q + \frac{2q^3}{3}\right]_{q=0}^2 dp = \int_0^3 \left(2p^2 + \frac{16}{3}\right) dp$$

$$= \left[\frac{2p^3}{3} + \frac{16}{3}p\right]_{p=0}^3 = (18 + 16) = 34$$

Recall $\vec{R}(p,q) = \left(\frac{1}{2}p^2, q^2, pq\right)$ for $0 \le p \le 3$ and $0 \le q \le 2$.

c. (10 points) Compute the mass of the surface, S, if the surface mass density is $\rho(x,y,z) = z$.
HINT: $M = \iint \rho\, dS$

$$M = \iint \rho \left|\vec{N}\right| dp\,dq = \int_0^2 \int_0^3 pq(p^2 + 2q^2)\,dp\,dq = \int_0^2 \left[\frac{p^4 q}{4} + p^2 q^3 \right]_{p=0}^{3} dq = \int_0^2 \left(\frac{81q}{4} + 9q^3 \right) dq$$

$$= \left[\frac{81q^2}{8} + \frac{9q^4}{4} \right]_{q=0}^{2} = \left(\frac{81}{2} + 36 \right) = \frac{153}{2} \qquad \text{OR}$$

$$M = \iint \rho \left|\vec{N}\right| dq\,dp = \int_0^3 \int_0^2 pq(p^2 + 2q^2)\,dq\,dp = \int_0^3 \left[\frac{p^3 q^2}{2} + \frac{2pq^4}{4} \right]_{q=0}^{2} dp = \int_0^3 (2p^3 + 8p)\,dp$$

$$= \left[\frac{p^4}{2} + 4p^2 \right]_{p=0}^{3} = \left(\frac{81}{2} + 36 \right) = \frac{153}{2}$$

d. (10 points) Compute the flux through the surface, S, of the vector field $\vec{F} = (2x, 2y, z)$ if the surface is oriented down and out.
HINT: $Flux = \iint \vec{F} \cdot d\vec{S}$

$\vec{N} = (-2q^2, -p^2, 2pq)$ is up and in. Reverse it. $\vec{N} = (2q^2, p^2, -2pq)$

$\vec{F} = (2x, 2y, z) = (p^2, 2q^2, pq)$

$$Flux = \iint \vec{F} \cdot \vec{N}\,dp\,dq = \int_0^2 \int_0^3 (2p^2q^2 + 2p^2q^2 - 2p^2q^2)\,dp\,dq = \int_0^2 \int_0^3 2p^2q^2\,dp\,dq$$

$$= 2\left[\frac{p^3}{3} \right]_0^3 \left[\frac{q^3}{3} \right]_0^2 = 2(9)\left(\frac{8}{3} \right) = 48$$

e. (6 points HONORS ONLY) Find the equation of the plane tangent to the surface, S, at the point where $(p,q) = (2,1)$.

$\vec{R}(p,q) = \left(\frac{1}{2}p^2, q^2, pq\right) \qquad \vec{N} = (-2q^2, -p^2, 2pq)$

$P = \vec{R}(2,1) = (2,1,2) \qquad \vec{N} = (-2,-4,4) \qquad \vec{N} \cdot X = \vec{N} \cdot P$

$-2x - 4y + 4z = -2(2) - 4(1) + 4(2) = -4 - 4 + 8 = 0$
$x + 2y - 2z = 0$