1-11	/66	13	/10
12	/20	14	/10
		Total	/106

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MATH 253 Exam 1 Fall 2009

Sections 501,503 Solutions P. Yasskin

Multiple Choice: (6 points each. No part credit.)

1. If $f(x,y) = x^2\cos(y^2)$, which of the following is FALSE?

a. $f_x(x,y) = 2x\cos(y^2)$

b. $f_y(x,y) = -2x^2y\sin(y^2)$

c. $f_{xx}(x,y) = 2\cos(y^2)$

d. $f_{yy}(x,y) = -4x^2y\cos(y^2)$ Correct Choice

e. $f_{xy}(x,y) = -4xy\sin(y^2)$

$$f_{yy}(x,y) = \frac{d}{dy}[-2x^2y\sin(y^2)] = -2x^2\sin(y^2) - 4x^2y^2\cos(y^2)$$

2. The quadratic surface $x^2 - y^2 + z^2 - 4x - 6y - 10z + 16 = 0$ is a

a. hyperboloid of 1 sheet and center $(2,3,5)$

b. hyperboloid of 1 sheet and center $(2,-3,5)$ Correct Choice

c. hyperboloid of 2 sheets and center $(2,3,5)$

d. hyperboloid of 2 sheets and center $(2,-3,5)$

e. cone with vertex $(2,3,5)$

$(x^2 - 4x) - (y^2 + 6y) + (z^2 - 10z) + 16 = 0$

$(x^2 - 4x + 4) - (y^2 + 6y + 9) + (z^2 - 10z + 25) + 16 = 4 - 9 + 25$

$(x-2)^2 - (y+3)^2 + (z-5)^2 = 4$ hyperboloid with center $(2,-3,5)$

$(x-2)^2 + (z-5)^2 = 4 + (y+3)^2$ $(x-2)^2 + (z-5)^2 \ge 4$ 1 sheet

3. An airplane is travelling due North at constant speed and a constant altitude as it crosses the equator. In what direction does the $\hat{B}$ vector point?
HINTS: Remember the Earth is curved. Ignore the rotation of the Earth.

a. East

b. West Correct Choice

c. South

d. Up

e. Down

$\hat{T}$ points North. $\hat{N}$ points Down. So $\hat{B} = \hat{T} \times \hat{N}$ points West.

4. A triangle has edge vectors $\overrightarrow{AB} = (2,1,-2)$ and $\overrightarrow{AC} = (-2,-2,4)$.
Find the altitude of the triangle if $\overline{AB}$ is the base.

a. $\dfrac{2\sqrt{5}}{3}$ Correct Choice

b. $\dfrac{\sqrt{5}}{3}$

c. $2\sqrt{5}$

d. $\sqrt{5}$

e. $3\sqrt{5}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & 1 & -2 \\ -2 & -2 & 4 \end{vmatrix} = \hat{\imath}(4-4) - \hat{\jmath}(8-4) + \hat{k}(-4+2) = (0,-4,-2)$$

$$Area = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{(-4)^2 + (-2)^2} = \frac{1}{2}\sqrt{20} = \sqrt{5}$$

$$Area = \frac{1}{2} Base \cdot Altitude \qquad Base = \left|\overrightarrow{AB}\right| = \sqrt{4+1+4} = 3 \qquad Altitude = \frac{2Area}{Base} = \frac{2\sqrt{5}}{3}$$

5. A box slides down the helical ramp $\vec{r}(t) = (4\cos t, 4\sin t, 9-3t)$ starting at height $z = 9$ and ending at height $z = 0$. How far does the box slide?

a. 3

b. 5

c. 15 Correct Choice

d. 25

e. 75

$$\vec{v} = (-4\sin t, 4\cos t, -3) \qquad |\vec{v}| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5$$

$$L = \int_{z=9}^{z=0} ds = \int_0^3 |\vec{v}|\,dt = \int_0^3 5\,dt = \Big[5t\Big]_0^3 = 15$$

6. A box slides down the helical ramp $\vec{r}(t) = (4\cos t, 4\sin t, 9-3t)$ starting at height $z = 9$ and ending at height $z = 0$ under the action of the force $\vec{F} = (-yz, xz, 5z)$.
Find the work done on the box.

a. $\dfrac{9}{2}$

b. 9

c. $\dfrac{25}{2}$

d. $\dfrac{27}{2}$ Correct Choice

e. 27

$$\vec{v} = (-4\sin t, 4\cos t, -3) \qquad \vec{F} = (-yz, xz, 5z) = (-4\sin t\,(9-3t), 4\cos t\,(9-3t), 5\,(9-3t))$$

$$\vec{F} \cdot \vec{v} = 16\sin^2 t\,(9-3t) + 16\cos^2 t\,(9-3t) - 15\,(9-3t) = 9-3t$$

$$W = \int_{z=9}^{z=0} \vec{F} \cdot d\vec{s} = \int_0^3 \vec{F} \cdot \vec{v}\,dt = \int_0^3 (9-3t)\,dt = \left[9t - \frac{3}{2}t^2\right]_0^3 = \frac{27}{2}$$

7. The diameter and height of a cylindrical trash can (no lid) are measured as $D = 30$ cm and $h = 40$ cm. The metal is 0.2 cm thick. Use differentials to estimate the volume of metal used to make the can.

a. 165π cm^3

b. 210π cm^3

c. 285π cm^3 Correct Choice

d. 330π cm^3

e. 525π cm^3

$V = \pi r^2 h \qquad r = 15 \qquad h = 40 \qquad dr = 0.2 \qquad dh = 0.2$

$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh = (2\pi rh)dr + (\pi r^2)dh = 2\pi \cdot 15 \cdot 40 \cdot 0.2 + \pi 15^2 \cdot 0.2 = 285\pi$

$V = \frac{1}{4}\pi D^2 h \qquad D = 30 \qquad h = 40 \qquad dD = 0.4 \qquad dh = 0.2$

$dV = \frac{\partial V}{\partial D}dD + \frac{\partial V}{\partial h}dh = \left(\frac{1}{2}\pi Dh\right)dD + \left(\frac{1}{4}\pi D^2\right)dh = \frac{1}{2}\pi \cdot 30 \cdot 40 \cdot 0.4 + \frac{1}{4}\pi 30^2 \cdot 0.2 = 285\pi$

8. Find the equation of the plane tangent to the surface $z = x^3 y^2$ at the point $(2,1)$. Then the z-intercept is $z =$

a. -40

b. 8

c. -8

d. 32

e. -32 Correct Choice

$f(x,y) = x^3y^2 \qquad f(2,1) = 8$ The tangent plane is

$f_x(x,y) = 3x^2y^2 \qquad f_x(2,1) = 12 \qquad z = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$

$f_y(x,y) = 2x^3y \qquad f_y(2,1) = 16 \qquad = 8 + 12(x-2) + 16(y-1)$

$= 12x + 15y - 32$

The intercept is $z = -32$.

9. Find the equation of the plane tangent to the surface $12xyz - z^3 = 45$ at the point $(1,2,3)$. Then the z-intercept is $z =$

a. 135

b. 45

c. $-\sqrt[3]{6}$

d. -45 Correct Choice

e. -135

$F(x,y,z) = xyz - z^3 \qquad \vec{\nabla}F = (12yz, 12xz, 12xy - 3z^2) \qquad \vec{N} = \left|\vec{\nabla}F\right|_{(1,2,3)} = (72, 36, -3)$

$P = (1,2,3) \qquad \vec{N} \cdot X = \vec{N} \cdot P \qquad 72x + 36y - 3z = 72 \cdot 1 + 36 \cdot 2 - 3 \cdot 3 = 135$

z-intercept when: $x = 0, y = 0, \qquad -3z = 135 \qquad z = -45$

10. Starting from the point $(1,-2)$, find the maximum rate at which the function $f(x,y) = x^2y^3$ increases.

a. 20 Correct Choice

b. 25

c. 400

d. $(-16,12)$

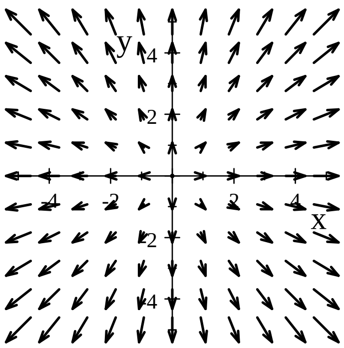
e. $(16,-12)$

$\vec{\nabla}f = (2xy^3, 3x^2y^2) \qquad \vec{\nabla}f(1,-2) = (-16,12)$

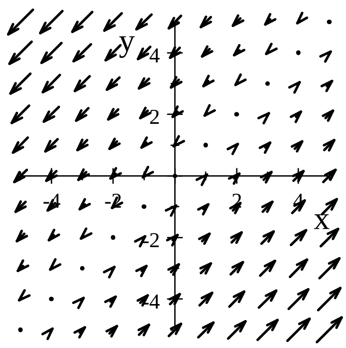
The maximum rate of increase is $\left|\vec{\nabla}f(1,-2)\right| = \sqrt{16^2+12^2} = \sqrt{256+144} = \sqrt{400} = 20$

11. Which of the following is the plot of the vector field $F(x,y) = (x+y, x-y)$?

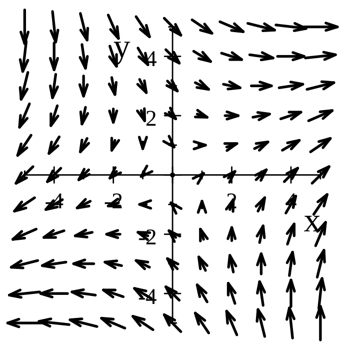
a.



d.

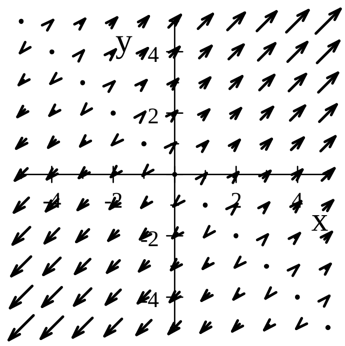


b.

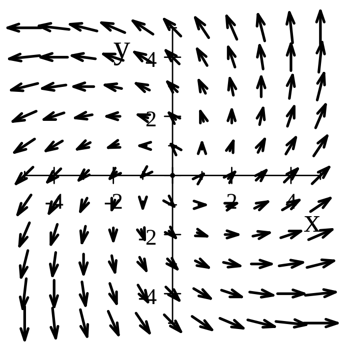


Correct Choice

e.



c.



When $y = x$, $F_2 = x - y = 0$ and $\vec{F}$ is horizontal.
When $y = -x$, $F_1 = x + y = 0$ and $\vec{F}$ is vertical.
So $\vec{F}$ is (b).

Work Out: (Points indicated. Part credit possible. Show all work.)

12. (20 points) Find the point on the curve $\vec{r}(t) = \left(e^t, \sqrt{2}\,t, e^{-t}\right)$ where the curvature is a local maximum or local minimum. Is it a local maximum or local minimum?

HINTS: First find the curvature $\kappa = \dfrac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3}$. Then find the critical point and apply the first or second derivative test.

$$\vec{v}(t) = \left(e^t, \sqrt{2}, -e^{-t}\right) \qquad |\vec{v}| = \sqrt{e^{2t}+2+e^{-2t}} = e^t + e^{-t}$$

$$\vec{a}(t) = (e^t, 0, e^{-t})$$

$$\vec{v}\times\vec{a} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ e^t & \sqrt{2} & -e^{-t} \\ e^t & 0 & e^{-t} \end{vmatrix} = \hat{\imath}\left(\sqrt{2}\,e^{-t}\right) - \hat{\jmath}(1+1) + \hat{k}\left(-\sqrt{2}\,e^t\right) = \left(\sqrt{2}\,e^{-t}, -2, -\sqrt{2}\,e^t\right)$$

$$|\vec{v}\times\vec{a}| = \sqrt{2e^{-2t}+4+2e^{2t}} = \sqrt{2}\sqrt{e^{-2t}+2+e^{2t}} = \sqrt{2}\,(e^t+e^{-t})$$

$$\kappa = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|^3} = \frac{\sqrt{2}\,(e^t+e^{-t})}{(e^t+e^{-t})^3} = \frac{\sqrt{2}}{(e^t+e^{-t})^2}$$

$$\kappa' = \frac{-2\sqrt{2}\,(e^t-e^{-t})}{(e^t+e^{-t})^3} = 0 \qquad e^t = e^{-t} \qquad e^{2t} = 1 \qquad t = 0$$

$$\vec{r}(0) = \left(e^0, \sqrt{2}\,0, e^{-0}\right) = \boxed{(1,0,1)}$$

$$\kappa'' = \frac{-2\sqrt{2}\,(e^t+e^{-t})^3(e^t+e^{-t}) + 2\sqrt{2}\,(e^t-e^{-t})3(e^t+e^{-t})^2(e^t-e^{-t})}{(e^t+e^{-t})^6}$$

$$= \frac{-2\sqrt{2}\,(e^t+e^{-t})^2 + 2\sqrt{2}\,(e^t-e^{-t})3(e^t-e^{-t})}{(e^t+e^{-t})^4}$$

$$\kappa''(0) = \frac{-2\sqrt{2}\,(1+1)^2 + 2\sqrt{2}\,(1-1)3(1-1)}{(1+1)^4} = \frac{-2\sqrt{2}}{4} < 0 \qquad \boxed{\text{local maximum}}$$

13. (10 points) The pressure, P, density, D, and temperature, T, of a certain ideal gas are related by $P = 4DT$. A fly is currently at the point $\vec{r}(t_0) = (3,2,4)$ and has velocity $\vec{v}(t_0) = (2,1,2)$.

At the point $(3,2,4)$, the density and temperature and their gradients are

$$D = 50 \qquad \vec{\nabla}D = \left(\frac{\partial D}{\partial x}, \frac{\partial D}{\partial y}, \frac{\partial D}{\partial z}\right) = (0.1, 0.4, 0.2)$$

$$T = 300 \qquad \vec{\nabla}T = \left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = (2,3,1)$$

Find the time rate of change of the pressure, $\dfrac{dP}{dt}$, as seen by the fly.

$$\frac{dP}{dt} = \frac{dP}{dD}\frac{dD}{dt} + \frac{dP}{dT}\frac{dT}{dt} = \frac{dP}{dD}\left(\frac{dD}{dx}\frac{dx}{dt} + \frac{dD}{dy}\frac{dy}{dt} + \frac{dD}{dz}\frac{dz}{dt}\right) + \frac{dP}{dT}\left(\frac{dT}{dx}\frac{dx}{dt} + \frac{dT}{dy}\frac{dy}{dt} + \frac{dT}{dz}\frac{dz}{dt}\right)$$

$$= 4T\left(\vec{\nabla}D\cdot\vec{v}\right) + 4D\left(\vec{\nabla}T\cdot\vec{v}\right) = 4\cdot 300\,(0.2+0.4+0.4) + 4\cdot 50\,(4+3+2) = 1200 + 1800 = 3000$$

14. (10 points) Suppose $p = p(x,y)$, while $x = x(u,v)$ and $y = y(u,v)$.

Further, you know the following information:

$x(2,1) = 3$	$y(2,1) = 4$	$p(2,1) = 13$	$p(3,4) = 14$
$\frac{\partial p}{\partial x}(2,1) = 5$	$\frac{\partial p}{\partial y}(2,1) = 6$	$\frac{\partial p}{\partial x}(3,4) = 7$	$\frac{\partial p}{\partial y}(3,4) = 8$
$\frac{\partial x}{\partial u}(2,1) = 9$	$\frac{\partial x}{\partial v}(2,1) = 10$	$\frac{\partial x}{\partial u}(3,4) = 11$	$\frac{\partial x}{\partial v}(3,4) = 12$
$\frac{\partial y}{\partial u}(2,1) = 18$	$\frac{\partial y}{\partial v}(2,1) = 17$	$\frac{\partial y}{\partial u}(3,4) = 16$	$\frac{\partial y}{\partial v}(3,4) = 15$

a. Write out the chain rule for $\frac{\partial p}{\partial u}$.

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial u}$$

b. Then use it and the above information to compute $\frac{\partial p}{\partial u}(2,1)$.

$$\frac{\partial p}{\partial u} = \frac{\partial p}{\partial x}\bigg|_{(x(u,v),y(u,v))}\frac{\partial x}{\partial u} + \frac{\partial p}{\partial y}\bigg|_{(x(u,v),y(u,v))}\frac{\partial y}{\partial u}$$

$$\frac{\partial p}{\partial u}(2,1) = \frac{\partial p}{\partial x}(3,4)\frac{\partial x}{\partial u}(2,1) + \frac{\partial p}{\partial y}(3,4)\frac{\partial y}{\partial u}(2,1) = 7 \cdot 9 + 8 \cdot 18 = 207$$